ELECTRICAL GENERATORS USED FOR WIND POWER GENERATION

The mechanical power of a wind turbine is converted into electric power by an alternating current (AC) generator or a direct current (DC) generator. The AC generator can be either a synchronous machine or an induction machine, the latter being the most widely used in the wind power industry. The electrical machine works on the principle of action and reaction of electromagnetic induction. The resulting electromechanical energy conversion is reversible. The same machine can be used as a motor for converting electric power into mechanical power or as a generator for converting mechanical power into electric power.

Figure 1 depicts common construction features of electrical machines. Typically, there is an outer stationary member (stator) and an inner rotating member (rotor). The rotor is mounted on bearings fixed to the stator. Both the stator and the rotor carry cylindrical iron cores, which are separated by an air gap. The cores are made of magnetic iron of high permeability and have conductors embedded in slots distributed on the core surface. Alternatively, the conductors are wrapped in the coil form around salient magnetic poles. Figure 2 is a cross-sectional view of a rotating electrical machine with the stator made of salient poles and the rotor with distributed conductors. The magnetic flux, created by the excitation current in one of the two coils, passes from one core to the other in a combined magnetic circuit always forming a closed loop. Electromechanical energy conversion is accomplished by interaction of the magnetic flux produced by one coil with the electrical current in the other coil. The current may be externally supplied or electromagnetically induced. The induced current in a coil is proportional to the rate of change in the flux linkage of that coil.

The various types of machines differ fundamentally in the distribution of the conductors forming the windings, and by their elements: whether they have continuous slotted cores or salient poles. The electrical operation of any given machine depends on the nature of the voltage applied to its windings. The narrow annular air gap between the stator and the rotor is the critical region of the machine operation, and the theory of performance is mainly concerned with the conditions in or near the air gap.
All machines in the internal working are AC machines, because the conductors rotate in the magnetic flux of alternate north and south poles. The DC machine converts the inside AC into DC for outside use. It does so by using a mechanical commutator. The commutator performs this function by sliding carbon brushes along a series of copper
segments. It switches the positive output terminal continuously to the conductor generating the positive polarity voltage, and likewise for the negative polarity terminal. The sliding contacts inherently result in low reliability and high maintenance cost. Despite this disadvantage, the DC machine was used extensively as a motor until the early 1980s because of its extremely easy speed control. It has been used as a generator in a limited number of wind power installations of small capacity, particularly where the electricity could be locally used in the DC form. However, the conventional DC machine with a mechanical commutator and sliding carbon brushes has fallen out of favor in the present day. Its brushless version is used where the DC machine gives a system advantage.

The conventional DC machine is either self-excited by shunt or by series coils carrying DC current to produce a magnetic field. The modern DC machine is often designed with permanent magnets to eliminate the field current requirement, hence, the commutator. It is designed in the “inside-out” configuration. The rotor carries the permanent-magnet poles and the stator carries the wound armature. The stator produces AC current, which is then rectified using the solid-state semi conducting devices. Such a machine does not need the commutator and the brushes; hence, the reliability is greatly improved. The permanent-magnet DC machine is used with small wind turbines. However, due to limitations of the permanent-magnet capacity and strength, the brushless DC machine is generally limited to ratings below 100 kW.

SYNCHRONOUS GENERATOR

The synchronous generator produces most of the electric power consumed in the world. For this reason, the synchronous machine is an established machine. The machine works at a constant speed related to the fixed supply frequency. Therefore, it is not well suited for variable-speed operation in wind plants without power electronic frequency converters. Moreover, the conventional synchronous machine requires DC current to excite the rotor field, which has traditionally used sliding carbon brushes on slip rings on the rotor shaft. This introduces some unreliability in its use. The modern synchronous machines are made brushless by generating the required DC field current on the rotor itself. Reliability is greatly improved while reducing the cost. The DC field current need
can be eliminated altogether by using a reluctance rotor, in which the synchronous operation is achieved by the reluctance torque. The reluctance machine rating, however, is limited to tens of kW. It is being investigated at present for small wind generators.

The synchronous machine, when used in a grid-connected system, has some advantages over the induction machine. It does not require reactive power from the grid. This results in a better quality of power at the grid interface. This advantage is more pronounced when the wind farm is connected to a small-capacity grid using a long low-voltage transmission link.

The synchronous generator is rarely used in gear-driven wind systems. However, the low-speed design of the synchronous generator is often found advantageous in the direct-drive variable-speed wind turbine. In such a design, the generator is completely decoupled from the grid by a voltage source power electronic converter connected to the stator, and the rotor is excited by an excitation winding or a permanent magnet.

**INDUCTION GENERATOR**

The electric power in industry is consumed primarily by induction machines working as motors driving mechanical loads. For this reason, the induction machine, invented by Nikola Tesla and financed by George Westinghouse in the late 1880s, represents a well-established technology. The primary advantage of the induction machine is the rugged brushless construction that does not need a separate DC field power. The disadvantages of both the DC machine and the synchronous machine are eliminated in the induction machine, resulting in low capital cost, low maintenance, and better transient performance. For these reasons, the induction generator is extensively used in small and large wind farms and small hydroelectric power plants. The machine is available in numerous power ratings up to several megawatts capacity, and even larger. For economy and reliability, many wind power systems use induction machines as electrical generators.

**CONSTRUCTION OF INDUCTION GENERATOR**

In the electromagnetic structure of the induction generator, the stator is made of numerous coils wound in three groups (phases), and is supplied with three-phase current.
The three coils are physically spread around the stator periphery and carry currents, which are out of time phase. This combination produces a rotating magnetic field, which is a key feature in the working of the induction machine. The angular speed of the rotating magnetic field is called the *synchronous speed*. It is denoted by $N_s$ and is given by the following in rpm

$$N_s = \frac{120f}{p}$$  \hspace{1cm} (I)

Where $f = $ frequency of the stator excitation
$p = $ Number of magnetic poles.

The stator coils are embedded in slots in a high-permeability magnetic core to produce the required magnetic field intensity with a small exciting current. The rotor, however, has a completely different structure. It is made of solid conducting bars, also embedded in slots in a magnetic core. The bars are connected together at both ends by two conducting end rings (Figure 3). Because of its resemblance, the rotor is called a *squirrel cage rotor*, or the *cage rotor*, for short, and the motor is called the *squirrel cage induction motor*.

![Figure 3](image_url) **FIGURE 3** Squirrel cage rotor of the induction machine under rotating magnetic field.

**WORKING PRINCIPLE**

The stator magnetic field is rotating at the synchronous speed determined by Equation 1. This field is conceptually represented by the rotating magnets in Figure 3.
The relative speed between the rotating field and the rotor induces the voltage in each closed loop of the rotor conductors linking the stator flux $\varphi$. The magnitude of the induced voltage is given by Faraday’s law of electromagnetic induction, namely:

$$ e = -\frac{d\varphi}{dt} \quad \text{(2)} $$

Where $\varphi$ = the magnetic flux of the stator linking the rotor loop.

This voltage in turn sets up the circulating current in the rotor. The electromagnetic interaction of the rotor current and stator flux produces the torque. The magnitude of this torque is given by the following:

$$ T = k\Phi I_2 \cos \phi_2 \quad \text{(3)} $$

where

- $k$ = constant of proportionality
- $\Phi$ = magnitude of the stator flux wave
- $I_2$ = magnitude of induced current in the rotor loops
- $\phi_2$ = phase angle by which the rotor current lags the rotor voltage

The rotor accelerates under this torque. If the rotor were on frictionless bearings in a vacuum with no mechanical load attached, it would be completely free to rotate with zero resistance. Under this condition, the rotor would attain the same speed as the stator field, namely, the synchronous speed. At this speed, the current induced in the rotor is zero, no torque is produced, and none is required. Under these conditions, the rotor finds equilibrium and will continue to run at the synchronous speed.

If the rotor is now attached to a mechanical load such as a fan, it will slow down. The stator flux, which always rotates at a constant synchronous speed, will have a relative speed with respect to the rotor. As a result, electromagnetically induced voltage, current, and torque are produced in the rotor. The torque produced must equal that needed to drive the load at this speed. The machine works as a motor in this condition.

If we attach the rotor to a wind turbine and drive it faster than its synchronous speed via a step-up gear, the induced current and the torque in the rotor reverse the direction. The machine now works as the generator, converting the mechanical power of the turbine into electric power, which is delivered to the load connected to the stator.
terminals. If the machine were connected to a grid, it would feed power into the grid. Thus, the induction machine can work as an electrical generator only at speeds higher than the synchronous speed. The generator operation, for this reason, is often called the *super synchronous operation* of the induction machine.

As described in the preceding text, an induction machine needs no electrical connection between the stator and the rotor. Its operation is entirely based on electromagnetic induction; hence, the name. The absence of rubbing electrical contacts and simplicity of its construction make the induction generator a very robust, reliable, and low-cost machine. For this reason, it is widely used in numerous industrial applications.

Engineers familiar with the theory and operation of the electrical transformer would see the working principle of the induction machine can be seen as the transformer with shorted secondary coil. The high-voltage coil on the stator is excited, and the low-voltage coil on the rotor is shorted on itself. The electrical or mechanical power from one to the other can flow in either direction. The theory and operation of the transformer, therefore, holds true when modified to account for the relative motion between the stator and the rotor. This motion is expressed in terms of the slip of the rotor relative to the synchronously rotating magnetic field.

**ROTOR SPEED AND SLIP**

The slip of the rotor is defined as the ratio of the speed of rotating magnetic field sweeping past the rotor and the synchronous speed of the stator magnetic field as follows:

\[ s = \frac{N_s - N_r}{N_s} \] (4)

where

- \( s \) = slip of the rotor in a fraction of the synchronous speed
- \( N_s \) = synchronous speed = 60 \( \text{f} \text{t}/\text{p} \)
- \( N_r \) = rotor speed

The slip is positive in the motoring mode and negative in the generating mode. In both modes, a higher rotor slip induces a proportionally higher current in the rotor, which results in greater electromechanical power conversion. In both modes, the value of slip is
generally a few to several percent. Higher slips, however, result in greater electrical loss, which must be effectively dissipated from the rotor to keep the operating temperature within the allowable limit.

The heat is removed from the machine by the fan blades attached to one end ring of the rotor. The fan is enclosed in a shroud at the end. The forced air travels axially along the machine exterior, which has fins to increase the dissipation area.

The induction generator feeding a 60-Hz grid must run at a speed higher than 3600 rpm in a 2-pole design, 1800 rpm in a 4-pole design, and 1200 rpm in a 6-pole design. The wind turbine speed, on the other hand, varies from a few hundred rpm in kW-range machines to a few tens of rpm in MW-range machines. The wind turbine, therefore, must interface the generator via a mechanical gear. As this somewhat degrades efficiency and reliability, many small stand-alone plants operate with custom-designed generators operating at lower speeds without any mechanical gear.

Under the steady-state operation at slip “s,” the induction generator has the following operating speeds in rpm:

- Stator flux wave speed: \( N_s \)
- Rotor mechanical speed: \( N_r = (1 - s)N_s \)
- Stator flux speed with respect to rotor: \( sN_s \)
- Rotor flux speed with respect to stator: \( N_r + sN_s = N_s \) (5)

Thus, the squirrel cage induction machine is essentially a constant-speed machine, which runs slightly slipping behind the rotating magnetic field of the three phase stator current. The rotor slip varies with the power converted, and the rotor speed variations are within a few percent. It always consumes reactive power — undesirable when connected to a weak grid — which is often compensated by capacitors to achieve the systems power factor closed to one. Changing the machine speed is difficult. It can be designed to run at two different but fixed speeds by changing the number of poles of the stator winding.

The voltage usually generated in the induction generator is 690-V AC. It is not economical to transfer power at such a low voltage over a long distance. Therefore, the machine voltage is stepped up to a higher value between 10,000 V and 30,000 V via a step-up transformer to reduce the power losses in the lines.
SELF EXCITATION OF INDUCTION GENERATOR (SEIG)

The IG with capacitor excitation is driven by a prime mover with the main power switch open (Figure 4(a)). As the speed increases, due to prime-mover torque, eventually, the no-load terminal voltage increases and settles to a certain value, depending on machine speed, capacitance, and machine parameters.

Figure (4) Self-excitation on self-excited induction generator (SEIG):
(a) The general scheme, (b) oversimplified equivalent circuit, and
(c) quasi-steady-state self-excitation characteristics.

The equivalent circuit is further simplified by neglecting the stator resistance and leakage inductance and by considering zero slip (s=0 open rotor circuit) for no-load conditions (Figure 4.3(b)). $E_{rem}$ represents the no-load initial stator voltage (before self-
excitation), at frequency at frequency \( \omega_{10} = \omega_r \), produced by the remnant flux density in the rotor left there from previous operation events.

To initiate the self-excitation process, \( E_{\text{rem}} \) has to be nonzero. The magnetization curve of the IG, obtained from typical motor no-load tests, \( E_1(I_m) \) has to advance to the nonlinear (saturation) zone in order to firmly intersect the capacitor straight-line voltage characteristic (Figure 4.(c)) and, thus, produce the no-load voltage \( E_1 \). The process of self-excitation of IG has been known for a long time.

The increasing of the terminal voltage from \( V_{\text{rem}} \) to \( V_{10} \) unfolds slowly in time (seconds), and Figure 4(c) presents it as a step-wise quasi-steady-state process. It is a qualitative representation only. Once the SEIG is self-excited, the load is connected. If the load is purely resistive, the terminal voltage decreases and so does (slightly) the frequency \( \omega_1 \) for constant (regulated) prime-mover speed \( \omega_r \).

With \( \omega_1 < \omega_r \), the SEIG delivers power to the load for negative slip \( S < 0 \)

\[
\frac{n}{f_1} = \frac{np}{1 + |S|} \quad S < 0
\]

The computation of terminal voltage \( V_1 \), frequency \( f_1 \), stator current \( I_1 \), delivered active and reactive power (efficiency) for given load (speed \( n \)), capacitor \( C \), and machine parameters, \( R_1, R_2, L_{11}, L_{22}, L_m(I_m) \) represents, in fact, the process of obtaining the steady-state performance. The nonlinear function \( L_m(I_m) \) magnetization curve — and the variation of frequency \( f_1 \) with load, at constant speed \( n \), make the process mathematically intricate.
Steady-State Analysis of Three-Phase SEIGs

Various analytical (and numerical) methods to calculate the steady-state performance of SEIGs were proposed. They seem, however, to fall into two main categories:

- Loop impedance models
- Nodal admittance models

Both models are based on the SEIG equivalent circuit (Figure 5) expressed in per unit (P.U.) form for frequency \( f \) (P.U.) and speed \( U \) (in P.U.) as follows:

\[
\frac{f}{f_{1b}} = \frac{f_{1}}{f_{1b}}; \quad \frac{U}{U_{1b}} = \frac{n p_{1}}{f_{1b}};
\]

The base frequency for which all reactances \( X_{1r}, X_{2r}, X_{m}(I_m) \) are calculated is \( f_{1b} \).

With an \( R_L, L_L, C_L \) load, the equivalent circuit in Figure 5 with speed and frequency in P.U. terms becomes as shown in Figure 6.
The presence of frequency $f$ in the load, the dependence of core loss resistance $R_m$ of frequency $f$, and the nonlinear dependence on $X_m$ of $I_m$ makes the solving of the equivalent circuit difficult. The SEIG plus load show zero total impedance:

$$R_e (IG+load) = 0$$
$$X_e (IG+excitation\_capacitor+load) = 0$$

for self-excitation, under load.

To solve it simply, the problem is reduced to two unknowns: $f$ (frequency) and $X_m$ for given excitation capacitor, IG ($R_1, R_2, X_{cl}, X_{c_l}, X_{cl}(I_m)$), load ($R_L, X_L, X_c$), and speed $U$.

- High-order polynomial equation (in $f$) approaches.
- Optimization approaches.

2.4.1 Second-Order Slip Equation Methods

The standard equivalent circuit of Figure 6 may be changed by lumping together the IG stator ($R_1, jfX_{cl}$), the excitation capacitor reactance ($-\frac{jX_c}{f}$), and the load ($R_L, jfX_L, -\frac{jX_{cl}}{f}$) into an equivalent series circuit ($R_{1L}, jfX_{1L}$). For self-excitation, $X_{1L} \leq 0$.
In general, both $R_{IL}$ and $X_{IL}$ are dependent on frequency $f$ (P.U.), though they get simplified forms if only a resistive or an $R_L X_L$ (an induction motor) is considered.

For self-excitation, the summation of currents in the node should be zero (with $E_i \neq 0$):

$$-I_1 + I_m - I_2 = 0$$

or

$$f E_i \left( \frac{1}{R_{IL} + jfX_{IL}} + \frac{S}{R_2 + jfX_{2l}} + \frac{1}{jfX_m} \right) = 0$$

The real and imaginary parts in Equation A have to be zero for self-excitation (it is, in fact, an energy balance condition):

$$\frac{R_{IL}}{R_{IL}^2 + f^2X_{IL}^2} + \frac{SR_2}{R_2^2 + f^2X_{2l}^2} = 0$$

$$\frac{1}{fX_m} - \frac{fX_{2l}}{R_{IL}^2 + f^2X_{IL}^2} + \frac{SfX_{2l}}{R_2^2 + f^2X_{2l}^2} = 0$$
For given frequency \( f \) (P.U.), Equation B remains (for given excitation capacitors, IG parameters, and load), with only one unknown, the slip \( S \):

\[
as^2 + bs + c = 0
\]

with

\[
a = f^2 X_{L}^2 R_{\text{L}}
\]

\[
b = R_2 \left( R_{\text{L}}^2 + f^2 X_{L}^2 \right)
\]

\[
c = R_2 R_2^2
\]

Equation C has two solutions, but only the smaller one (in amplitude) is really useful. For the larger one, most of the power is consumed into the rotor resistance:

\[
S_{\text{L}} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]

If complex solutions of \( S \) are obtained, it means that self-excitation is impossible.

With slip \( S \), found from Equation D, for given \( f \), the corresponding P.U. speed \( U = f - S \) is determined. With \( S = S_{\text{L}} \), from Equation B2, the magnetization reactance \( X_m \) is calculated as follows:

\[
X_m = f^{-1} \left[ \frac{f X_{L}}{R_2^2 + f^2 X_{L}^2} - \frac{S f X_{L}}{R_2^2 + S^2 f^2 X_{L}^2} \right] < X_{\text{max}}
\]

\( X_{\text{max}} \) is the maximum (unsaturated) value of the magnetization reactance (at base frequency \( f_b \)). With \( X_m > X_{\text{max}} \), self-excitation is again impossible.
**Arbitrary Reference Frame Theory**

Synchronous and induction machine inductances are functions of the rotor speed, therefore the coefficients of the differential equations (voltage equations) which describe the behavior of these machines are time-varying. A change of variables can be used to reduce the complexity of machine differential equations, and represent these equations in another reference frame with constant coefficients. A change of variables which formulates a transformation of the 3-phase variables of stationary circuit elements to the arbitrary reference frame may be expressed as

\[
\mathbf{f}_{qbs} = K_s \mathbf{f}_{qbs},
\]

where,

\[
(\mathbf{f}_{qbs})^T = \begin{bmatrix} f_{qs} & f_{ds} & f_{os} \end{bmatrix},
\]

\[
(\mathbf{f}_{qbs})^T = \begin{bmatrix} f_{os} & f_{bs} & f_{cs} \end{bmatrix},
\]

\[
K_s = \frac{2}{3} \begin{bmatrix}
\cos \theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\
\sin \theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix},
\]

\[
\theta = \int_0^t \omega(t) dt + \theta(0).
\]

\[
(K_s)^T = \begin{bmatrix}
\cos \theta & \sin \theta & 1 \\
\cos(\theta - \frac{2\pi}{3}) & \sin(\theta - \frac{2\pi}{3}) & 1 \\
\cos(\theta + \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) & 1
\end{bmatrix}.
\]
“f” can represent either voltage, current, or flux linkage. “s” indicates the variables, parameters and transformation associated with stationary circuits. “ω” represent the speed of reference frame $ω=0$. Stationary reference frame $ω=ω_r$: synchronously rotating reference frame $ω=ω_r$: rotor reference frame (i.e., the reference frame is fixed on the rotor) $f_{ds}, f_{qs}$ and $f_s$ may be thought of as the direction of the magnetic axes of the stator windings. $f_{qs}$ and $f_{ds}$ can be considered as the direction of the magnetic axes of the “new” fictitious windings located on $qs$ and $ds$ axis which are created by the change of variables. Power Equations corresponding to the transformations is given by

$$P_{abc} = V_{ab}J_{ab} + V_{bc}J_{bc} + V_{ca}J_{ca}$$

$$P_{qabc} = P_{abc} = \frac{3}{2}(V'_{q}J_{qs} + V'_{d}J_{ds} + 2V'_{r}J_{rs})$$

**Stationary circuit variables transformed to the arbitrary reference frame**

**Resistive elements:** For a 3-phase resistive circuit

$$V_{a} = \bar{r}i$$

$$V_{a} = (K_r)^{-1}V_{a}$$

$$L_{a} = (K_L)^{-1}L_{a}$$

$$V'_{a} = (K_r)^{-1}\bar{r}i$$

Inductive elements: For a 3-phase inductive circuit

$$V_{a} = \mu L_{a}$$

where, $\mu = \frac{dL}{dt}$,

$$\dot{L}_{a} = L_{a}$$

$$\mu L_{a} = \mu$$

$$\mu L_{a} = \mu$$

$$\mu L_{a} = \mu$$

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In terms of the substitute variables, we have

\[ \mathbf{V}_{qds} = \mathbf{K}_s \cdot p \left[ \mathbf{K}^{-1}_s \lambda_{qds} \right] = \mathbf{K}_s \cdot p \left[ \mathbf{K}^{-1}_s \right] \lambda_{qds} + \mathbf{K}_s \cdot \left[ \mathbf{K}^{-1}_s \right] p \lambda_{qds} \]

where, \( p \left[ \mathbf{K}^{-1}_s \right] = \omega \cdot \begin{bmatrix} -\sin \theta & \cos \theta & 0 \\ -\sin \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta - \frac{2\pi}{3} \right) & 0 \\ -\sin \left( \theta + \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) & 0 \end{bmatrix} \)

\[ \mathbf{K}_s \cdot p \left[ \mathbf{K}^{-1}_s \right] = \omega \cdot \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

\[ \mathbf{V}_{qds} = \mathbf{K}_s p \left( \mathbf{K}_s^{-1} \right) \lambda_{qds} + \mathbf{K}_s \left( \mathbf{K}_s^{-1} \right) p \lambda_{qds} \]

\[ \mathbf{V}_{qds} = \omega \lambda_{qds} + p \lambda_{qds} \]

where, \( \begin{bmatrix} \lambda_{qds} \end{bmatrix} = \begin{bmatrix} \lambda_{qb} & -\lambda_{qs} & 0 \end{bmatrix} \)

Vector equation \( \mathbf{V}_{qds} \) can be expressed as

\[ V_q = \omega \lambda_{qs} + p \lambda_{qs} \]
\[ V_d = -\omega \lambda_{qs} + p \lambda_{qs} \]
\[ V_s = p \lambda_{qs} \]

Where “\( \omega \lambda_{ds} \)” term and “\( \omega \lambda_{qs} \)” term are referred to as a “speed voltage” with the speed being the angular velocity of the arbitrary reference frame. When the reference frame is fixed in the stator, that is, the stationary reference frame (\( \omega = 0 \)), the voltage equations for the three-phase circuit become the familiar time rate of change of flux linkage in abc reference frame. For the three-phase circuit shown, \( \mathbf{L}_s \) is a diagonal matrix, and

\[ \lambda_{qds} = \mathbf{L}_s \lambda_{qds} \]
\[ \lambda_{qds} = \mathbf{K}_s \mathbf{L}_s \mathbf{K}_s^{-1} \lambda_{qds} = \mathbf{L}_s \lambda_{qds} \]
For the three-phase induction or synchronous machine, Ls matrix is expressed as

\[
\mathbf{L}_s = \begin{bmatrix}
I_{js} + I_{ms} & -\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} \\
-\frac{1}{2}I_{ms} & I_{js} + I_{ms} & -\frac{1}{2}I_{ms} \\
-\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} & I_{js} + I_{ms}
\end{bmatrix}
\]

Where, \( L_k \): leakage inductance, \( L_{ms} \): magnetizing inductance

\[
\mathbf{K}_s \mathbf{L}_s \left( \mathbf{K}_s \right)^{-1} = \\
\begin{bmatrix}
I_{js} + \frac{3}{2}I_{ms} & 0 & 0 \\
0 & I_{js} + \frac{3}{2}I_{ms} & 0 \\
0 & 0 & I_{js} + \frac{3}{2}I_{ms}
\end{bmatrix}
\]

**Inductive elements:** For a 3-phase inductive circuit with mutual inductance

Consider the stator windings of a symmetrical induction or round rotor synchronous machine shown below

\[
\mathbf{r}_s = \text{diag}[r_s, r_s, r_s]
\]

\[
\mathbf{L}_s = \begin{bmatrix}
I_s & M & M \\
M & I_s & M \\
M & M & I_s
\end{bmatrix}
\]

\[
I_s = I_{js} + I_{ms}
\]

\[
M = \frac{1}{2} I_{ms}
\]

For each phase voltage, we write the following equations,

\[
V_{as} = r_s i_{as} + p\lambda_{as}
\]

\[
V_{bs} = r_s i_{bs} + p\lambda_{bs}
\]

\[
V_{cs} = r_s i_{cs} + p\lambda_{cs}
\]

\[
V_{qdc\alpha} = \mathbf{K}_s V_{qdc\alpha}
\]

\[
i_{a\alpha} = \mathbf{K}_s i_{a\alpha}
\]

\[
\lambda_{a\alpha} = \mathbf{K}_s \lambda_{a\alpha}
\]

\[
\lambda_{a\beta\gamma} = \mathbf{L}_s \lambda_{a\beta\gamma}
\]

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In vector form,
\[ V_{abc} = r_j i_{abc} + p \lambda_{abc}, \]

Multiplying by \( K_S \)
\[ K_S V_{abc} = K_S r_j i_{abc} + K_S p \lambda_{abc} \]

Replace \( i_{abc} \) and \( \lambda_{abc} \) using the transformation equations
\[ K_S V_{abc} = K_S (r_j (K_S^{-1} i_{qds}) + K_S p (K_S^{-1} \lambda_{qds})) \]

\[ V_{qds} = r_j i_{qds} + \bar{\omega} \lambda_{qds} \]

Or
\[ V_{qs} = r_s i_{qs} + \omega \lambda_{ds} + p \lambda_{qs} \]
\[ V_{ds} = r_s i_{ds} - \omega \lambda_{qs} + p \lambda_{ds} \]
\[ V_{os} = r_s i_{os} + p \lambda_{os} \]

where, \( \bar{\omega} = \omega \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]
\[ \lambda_{qs} = (L_s - M) i_{qs} \]
\[ \lambda_{ds} = (L_s - M) i_{ds} \]
\[ \lambda_{os} = (L_s + 2M) i_{os} \]

Our equivalent circuit in arbitrary reference frame can be represented as

\( \omega \) = unspecified: stationary circuit variables referred to the arbitrary reference frame.

The variables are referred to as \( f_{qds}, \) \( f_{qs}, \) \( f_{ds} \) and \( f_{os} \) and transformation matrix is designated as \( K_S. \) \( \omega = 0: \) stationary circuit variables referred to the stationary reference frame. The variables are referred to as \( f_{qds}, \) \( f_{qs}, \) \( f_{ds} \) and \( f_{os} \) and transformation matrix is designated as \( K_S. \) \( \omega = \bar{\omega}: \) stationary circuit variables referred to the reference frame fixed in the rotor. The variables are referred to as \( f_{qds}, \) \( f_{qs}, \) \( f_{ds} \) and
\text{\(f_\text{qs}\)} and transformation matrix is designated as \(K_s\). \(\omega = \omega_e\): stationary circuit variables referred to the synchronously rotating reference frame. The variables are referred to as \(f_{\text{qs}}\), \(f_{\text{qs}}\), \(f_{\text{qs}}\), and \(f_{\text{qs}}\), and transformation matrix is designated as \(K_s\).

**Representation**

\[
\begin{align*}
\mathbf{f}_{\text{qs}}^s &\quad \text{Stationary reference frame} \\
\mathbf{f}_{\text{qs}}^r &\quad \text{Reference frame fixed on the rotor with speed of } \omega_r \\
\mathbf{f}_{\text{qs}}^e &\quad \text{Synchronously rotating reference frame}
\end{align*}
\]

\[
\begin{align*}
\mathbf{f}_{\text{qs}}^s &\quad \text{q-d axes of stator variables} \\
\mathbf{f}_{\text{qs}}^r &\quad \text{q-d axes of stator variables } \theta_r = \int_0^t \omega_r(t) dt \\
\mathbf{f}_{\text{qs}}^e &\quad \text{q-d axes of stator variables } \theta_e = \int_0^t \omega_e(t) dt
\end{align*}
\]

**Transformation of a Balanced Set**

Consider a 3-phase circuit which is excited by a balanced 3-phase voltage set. Assume the balanced set is a set of equal amplitude sinusoidal quantities which are displaced by 120°.

\[
\begin{align*}
f_{\text{qs}} &= \sqrt{2} f_s \cos \theta_{e}\_f \\
f_{\text{qs}} &= \sqrt{2} f_s \cos \left(\theta_{e}\_f - \frac{2\pi}{3}\right) \\
f_{\text{qs}} &= \sqrt{2} f_s \cos \left(\theta_{e}\_f + \frac{2\pi}{3}\right)
\end{align*}
\]

\(\theta_{e}\_f\): Angular position of each electrical variable (voltage, current, and flux linkage) is \(\theta_{e}\_f\) with the \(f\) subscript used to denote the specific electrical variable.

\(\theta_{e}\): Angular position of the synchronously rotating reference frame is \(\theta_{e}\). \(\theta_{e}\) and \(\theta_{e}\) differ only in the zero position \(\theta_{e}(0)\) and \(\theta_{e}(0)\), since each has the same angular velocity of \(\omega_e\), \(f_{\text{qs}}\), \(f_{\text{qs}}\), and \(f_{\text{qs}}\) can be transformed to the arbitrary reference frame,

\[
\mathbf{f}_{\text{qs}}^s = K_s \mathbf{f}_{\text{qs}}^e
\]

After transformation, we will have,

\[
\begin{align*}
f_{\text{qs}} &= \sqrt{2} f_s \cos (\theta_{e}\_f - \theta) \\
f_{\text{qs}} &= -\sqrt{2} f_s \sin (\theta_{e}\_f - \theta) \\
f_{\text{qs}} &= 0
\end{align*}
\]
qs and ds variables form a balanced 2-phase set in all reference frames except when \( \omega = \omega_c \):

\[
\begin{align*}
\begin{align*}
f_{qs}^e &= \sqrt{2} f_s \cos \left[ \theta_{qs}(0) - \theta_c(0) \right] \\
f_{ds}^e &= -\sqrt{2} f_s \sin \left[ \theta_{ds}(0) - \theta_c(0) \right]
\end{align*}
\]

In qs\(^e\) and ds\(^e\) reference frame, sinusoidal quantities appear as constant dc quantities.

**Balanced Steady-State Phasor Relationships**

For balanced steady-state conditions \( \omega_c \) is constant and sinusoidal quantities can be represented as phasor variables.

\[
\begin{align*}
F_{qs} &= \sqrt{2} F_s \cos \left[ \omega_c t + \theta_{qs}(0) \right] = \text{Re} \left[ \sqrt{2} F_s e^{j \theta_{qs}(0)} e^{j \omega_c t} \right] \\
F_{ds} &= \sqrt{2} F_s \cos \left[ \omega_c t + \theta_{ds}(0) - \frac{2\pi}{3} \right] = \text{Re} \left[ \sqrt{2} F_s e^{j \left( \theta_{ds}(0) - \frac{2\pi}{3} \right)} e^{j \omega_c t} \right] \\
F_{bs} &= \sqrt{2} F_s \cos \left[ \omega_c t + \theta_{bs}(0) + \frac{2\pi}{3} \right] = \text{Re} \left[ \sqrt{2} F_s e^{j \left( \theta_{bs}(0) + \frac{2\pi}{3} \right)} e^{j \omega_c t} \right]
\end{align*}
\]

Balanced steady-state qs-ds variables are,

\[
\begin{align*}
F_{qs}^e &= \sqrt{2} F_s \cos \left[ (\omega_c - \omega) t + \theta_{qs}(0) - \theta(0) \right] \\
&= \text{Re} \left[ \sqrt{2} F_s e^{j \left( \theta_{qs}(0) - \theta(0) \right)} e^{j \omega_c t} \right]
\end{align*}
\]

\[
\begin{align*}
F_{ds}^e &= -\sqrt{2} F_s \sin \left[ (\omega_c - \omega) t + \theta_{ds}(0) - \theta(0) \right] \\
&= \text{Re} \left[ \sqrt{2} F_s e^{j \left( \theta_{ds}(0) - \theta(0) \right)} e^{j \omega_c t} \right]
\end{align*}
\]

f\(_{as}\) phasor can be expressed as

\[
\vec{F}_{as} = F_{as} e^{j \theta_{as}(0)}
\]

For arbitrary reference frame \( (\alpha \neq \omega_c) \),

\[
\begin{align*}
\vec{F}_{qs} &= F_s e^{j \left( \theta_{qs}(0) - \theta(0) \right)} \\
\vec{F}_{ds} &= j F_{qs} \\
\end{align*}
\]

Selecting \( \theta(0) = 0 \),

\[
\begin{align*}
\vec{F}_{as} &= \vec{F}_{qs}
\end{align*}
\]

Thus, in all asynchronously rotating reference frame \( (\alpha \neq \omega_c) \) with \( \theta(0) = 0 \), the phasor representing the as variables is equal to the phasor representing the qs variables. In the synchronously rotating reference frame \( \omega = \omega_c \), \( F_{qs}^e \) and \( F_{ds}^e \) can be expressed as

\[
\begin{align*}
F_{qs}^e &= \text{Re} \left[ \sqrt{2} F_s e^{j \left( \theta_{qs}(0) - \theta(0) \right)} \right] \\
F_{ds}^e &= \text{Re} \left[ j \sqrt{2} F_s e^{j \left( \theta_{ds}(0) - \theta(0) \right)} \right]
\end{align*}
\]
Let $\theta_d(0)=0$, then

\[
F^e_{qs} = \sqrt{2} F_s \cos(\theta_{sf}(0)), \quad F^e_{ds} = -\sqrt{2} F_s \sin(\theta_{sf}(0))
\]

\[
\sqrt{2} F_{qs}^e = F^e_{qs} = jF^e_{ds}
\]

since, \[
\tilde{F}^e_{qs} = F^e_s e^{j\theta_{sf}(0)} = F^e_s \cos(\theta_{sf}(0)) + jF^e_s \sin(\theta_{sf}(0))
\]
**Transformations to Stationary Reference Frames:**

The applied approach eliminates the redundancy of poly phase winding substituting these by their two phase equivalent. Such redundancy is given for both line voltages and line currents of three phase ac machines. In a three phase machine with isolated neutral point (as shown in figure 1) the sum of line voltages and line currents has to be zero at all times according to kirchoff’s laws

![Diagram](image)

Figure 1 Three phase machine with isolated neutral point.

\[ i_a + i_b + i_c = 0 \] ..........................(1)

\[ U_{ab} + U_{bc} + U_{ca} = 0 \] ..........................(2)

According to (1) and (2) the input of a three phase ac motor is completely described by two line voltages and two currents since the third value equals the negative sum of the two others. Thus when describing the dynamic behavior of ac machines, the polyphase motor windings can be reduced to a set of two windings. The most suitable choice of describing the behavior of by a two axes equivalent is a set of field coils, having their magnetic axes arranged in quadrature. When saturation is neglected this approach is eliminates the mutual magnetic coupling of the three phase windings. In two axes equivalent, the flux of one winding does not interact with the perpendicular winding and vice versa.

The simplified diagram of a three phase ac motor (Fig 2) shows only the stator windings for each phase displaced by 120 degrees in space. The same is valid for rotor also. The transformation of stator and rotor variables to a two phase reference frame (indicated by subscripts q and d), where the coils are perpendicular, guarantees that there
is no interaction between perpendicular windings as there is no saturation, referred to as cross saturation.

Fig 2. Three Phase Winding and Two phase equivalent

Due to redundancy in (1)-(2), it is obvious that voltage and current equations of any ac machine can be reduced to a set of each two appropriate variables of q and d reference frame. One straightforward, but unusual option of transferring variables to the qd reference frame is a simple vector addition of the three phase variables. According to figure 2, a geometrical calculation yields

\[
\begin{bmatrix}
i_d, i_b \cos(120), i_c \cos(240)\end{bmatrix} = \frac{i_q}{(\text{Vector Addition})} \ldots \ldots (3)
\]

Solving the trigonometric functions and replacing \(i_c\) according to (1) results in

\[
\Rightarrow i_q = \frac{i_d - i_b}{2} \Rightarrow \frac{(-i_d - i_b)}{2} = i_q \ldots \ldots (4)
\]

\[
\Rightarrow i_q = \frac{3}{2} i_p \ldots \ldots (5)
\]

And similarly for d axis.
\[
\begin{align*}
\begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix}
\cos 30^\circ + i_c 
\cos 50^\circ
\end{bmatrix}
&= i_d \\
\Rightarrow \frac{\sqrt{3}}{2}i_b - \frac{\sqrt{3}}{2}(-i_a - i_b)
&= i_d \\
\Rightarrow i_d &= \frac{\sqrt{3}}{2}i_a + \frac{\sqrt{3}}{2}i_b
\end{align*}
\]

However when the above transformation is equally applied to both phase current and voltages, the system is not power invariant:

\[
p(t) = u_d j_a + u_b j_b + u_c j_c + u_d j_d + u_c j_c
\]

\[
u_d j_d + u_c j_d = \frac{3}{2} u_d \frac{3}{2} i_a + \left( \frac{\sqrt{3}}{2} u_a + \sqrt{3} u_b \right) \left( \frac{\sqrt{3}}{2} i_a + \sqrt{3} i_b \right)
\]

\[
= 3u_d j_a + 3u_b j_b + \frac{3}{2} u_d j_d
\]

\[
= \frac{3}{2} [u_d j_a + u_b j_b + (-u_a - u_b) (-i_a - i_b)] \\
= \frac{3}{2} [u_d j_a + u_b j_b + u_c j_c] = \frac{3}{2} p(t)
\]

Note, that the variables in the qd reference frame are no real values. They are fictitious and well chosen to describe the real electrical and dynamic behavior of the machine. Obviously, the original current phasor can be equally described by two perpendicular vector components multiplied by a constant scaling factor. According to (10) the power in the qd reference frame reduced by a scaling factor \(\frac{2}{3}\) guarantees power invariance.

Thus this scaling factor can be distributed amongst qd current and voltages in different ways. E.g. qd current remains and qd voltages are reduced by a factor of \(\frac{2}{3}\) or vice versa.

Therefore the final expression after power invariance is given by, from (1) and (6)

\[
\begin{bmatrix}
i_d \\
i_a
\end{bmatrix}
\begin{bmatrix}
1 & -\frac{1}{2} \\
0 & \frac{\sqrt{3}}{2}
\end{bmatrix}
\begin{bmatrix}
i_a \\
i_b \\
i_c
\end{bmatrix}
\]

\[
= \frac{3}{2} i_d \quad (11)
\]

\[
K_q j_{dc}
\]

\(K_q\) is known as transformation matrix

The above expression is valid for voltages, flux or flux linkages etc.
Unfortunately, while the sum of line voltages is zero, the phase voltages or phase to neutral voltages may contain a homopolar component dependent on the PWM method used. The effect of homopolar component on the machine with isolated neutral point is equal to the change of reference potential (e.g. changing the potential $u_o$ in the figure (1)). In other words the potential of all the three phases is varied uniformly and simultaneously. Fortunately, the sum of line voltages is always zero. Thus this homopolar component is unseen by the motor terminals and therefore not reflected in the motor behavior. However, since the motor voltages are often calculated by means of reference voltages containing such homopolar component, a more elaborate transformation must be used for the qd voltage calculation. The transformation matrix is obtained similarly to above method by geometrical calculations without replacing the phase voltages $u_c$. For completeness the zero components are also given. According to fig (3) the zero components indicates a displacement of reference potential $u_o$ with respect to the center $m$ of the line voltages. Obviously, the phase voltages $(u_a, u_b, u_c)$ are equal to the affiliated line to neutral voltages $(u_{am}, u_{bm}, u_{cm})$ plus the zero component $u_{mo}$. Since the sum of line to neutral voltages is zero, following relation is valid.

\[
U_{am} + U_{bm} + U_{cm} = (U_a - U_{mo}) + (U_b - U_{mo}) + (U_c - U_{mo}) = 0 \quad \ldots \ldots \quad (12)
\]

\[
\Rightarrow U_{mo} = \frac{1}{3}(u_a + u_b + u_c)
\]

Fig (3) Zero component of Phase Voltages: **Left:** Phase and Line Voltages without zero component **Right:** Phase voltages and displacement of the neutral point
Scaling of power in variance factor yields the zero the zero component in the qd reference frame.

\[
\begin{bmatrix}
i_q \\
i_d \\
i_o
\end{bmatrix} = \frac{2}{3} \begin{bmatrix}
1 & \frac{1}{\sqrt{3}} & -\frac{1}{2}
0 & \frac{\sqrt{3}}{2} & \frac{1}{2}
\frac{1}{2} & 1 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
i_s \\
i_c
\end{bmatrix} 
\]  
(12)

\[i_{qd} = K_s i_{dc}\]

\[K_s\] is known as transformation matrix.

The inverse transformation is

\[
\begin{bmatrix}
i_q \\
i_d \\
i_o
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & 0 & 1
-\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & 1
\frac{2}{1} & -\frac{2}{1} & 1
\end{bmatrix}
\begin{bmatrix}
i_s \\
i_c
\end{bmatrix} 
\]  
(13)

\[i_{dc} = K_s^{-1} i_{qd}\]

\[K_s^{-1}\] is known as inverse transformation matrix.

The above transformation matrix can be written in trigonometric functions as

\[
K_s = \frac{2}{3}
\begin{bmatrix}
\cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\
\sin\theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\]
3-Phase Induction Machines - Dynamic Modeling Using Reference Frame Theory

Winding arrangement for a 2-pole, 3-phase, wye-connected symmetrical induction machine is shown in Fig.1. Stator windings are identical, sinusoidally distributed windings, displaced by 120°, with \( N_s \) equivalent turns and resistance \( r_s \). Consider the case when rotor windings are also three identical sinusoidally distributed windings, displaced by 120°, with \( N_r \) equivalent turns and resistance \( r_r \).

![Diagram of 2 pole Three Phase Induction Machine](image)

In abc reference frame, voltage equations can be written as

\[
\begin{align*}
V_{abcd} &= r_s i_{abcd} + p \lambda_{abcd} \\
V_{abcr} &= r_r i_{abcr} + p \lambda_{abcr}
\end{align*}
\]

\[
(f_{abcd})^T = \begin{bmatrix} f_{as} & f_{bs} & f_{cs} \end{bmatrix}, \quad (f_{abcr})^T = \begin{bmatrix} f_{ar} & f_{br} & f_{cr} \end{bmatrix}
\]

\( s \): denotes variables and parameters associated with the stator circuits and \( r \): denotes variables and parameters associated with the rotor circuits.
$$\begin{bmatrix}
L_{\text{obs}} \\
L_{\text{obsr}}
\end{bmatrix} =
\begin{bmatrix}
I_x & I_{ys} & I_{ysr} \\
I_{ys} & -\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} \\
-\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} & I_{ys} + I_{ms}
\end{bmatrix}$$

Where,

$$L_s = \begin{bmatrix}
-I_{ys} + I_{ms} & -\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} \\
-\frac{1}{2}I_{ms} & I_{ys} + I_{ms} & -\frac{1}{2}I_{ms} \\
-\frac{1}{2}I_{ms} & -\frac{1}{2}I_{ms} & I_{ys} + I_{ms}
\end{bmatrix}, \quad L_r = \begin{bmatrix}
I_{ys} + I_{ms} & -\frac{1}{2}I_{mr} & -\frac{1}{2}I_{mr} \\
-\frac{1}{2}I_{mr} & I_{ys} + I_{ms} & -\frac{1}{2}I_{mr} \\
-\frac{1}{2}I_{mr} & -\frac{1}{2}I_{mr} & I_{ys} + I_{ms}
\end{bmatrix}$$

$L_{ys}$ and $L_{ms}$ are, respectively, the leakage and magnetizing inductance of the stator windings. $L_r$ and $L_{mr}$ are, respectively, the leakage and magnetizing inductance of the rotor windings.

$$L_{ys} = L_{rs} = I_{ms} \begin{bmatrix}
\cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\
\cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) \\
\cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r
\end{bmatrix}$$

“$L_{sr}$” is the amplitude of the mutual inductances between stator and rotor windings. A majority of induction machines are not equipped with coil-wound rotor windings; instead, the current flows in copper or aluminum bars which are uniformly distributed in a common ring at each end of the rotor. This type of rotor is referred to as a squirrel-cage rotor. Rotor variables can be referred to the stator windings by appropriate turn’s ratio.

$$I_{\text{obsr}} = \frac{N_s}{N_r} I_{\text{obs}}, \quad V'_{\text{obsr}} = \frac{N_s}{N_r} V_{\text{obs}}, \quad \lambda'_{\text{obsr}} = \frac{N_s}{N_r} \lambda_{\text{obs}}, \quad I_{\text{ms}} = \left(\frac{N_s}{N_r}\right)^2 I_{\text{sr}}$$

$$[L'_{sr}] = \frac{N_s}{N_r} [L_{sr}] = \begin{bmatrix}
\cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) \\
\cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r & \cos(\theta_r + \frac{2\pi}{3}) \\
\cos(\theta_r + \frac{2\pi}{3}) & \cos(\theta_r - \frac{2\pi}{3}) & \cos\theta_r
\end{bmatrix}$$

Also,

$$J_{mr} = \left(\frac{N_s}{N_r}\right)^2 J_{mr} \quad [L'_{r}] = \left(\frac{N_s}{N_r}\right)^2 [L_r]$$
\[
\begin{bmatrix}
I_{sr}' + I_{ms}' \\
\frac{1}{2}I_{ms}' \\
\frac{1}{2}I_{ms}'
\end{bmatrix}
= 
\begin{bmatrix}
I_{sr}' + I_{ms}' & -\frac{1}{2}I_{ms}' & -\frac{1}{2}I_{ms}' \\
-\frac{1}{2}I_{ms}' & I_{sr}' + I_{ms}' & -\frac{1}{2}I_{ms}' \\
-\frac{1}{2}I_{ms}' & -\frac{1}{2}I_{ms}' & I_{sr}' + I_{ms}'
\end{bmatrix}
\]

Where,
\[
I_{sr}' = \left(\frac{N_s}{N_f}\right)^2 I_{sr}
\]

Flux linkage may be expressed as
\[
\begin{bmatrix}
\lambda_{dcs}' \\
\lambda_{qcs}'
\end{bmatrix} =
\begin{bmatrix}
L_{s} & L_{sr}' \\
(I_{sr}')^T & L_{r}'
\end{bmatrix}
\begin{bmatrix}
\lambda_{dcs} \\
\lambda_{qcs}'
\end{bmatrix}
\]

Voltage equations expressed in terms of machine variables referred to the stator windings may be written as
\[
\begin{bmatrix}
V_{dcs}' \\
V_{qcs}'
\end{bmatrix} =
\begin{bmatrix}
r_s + pL_{s} & pL_{sr}' \\
p(I_{sr}')^T & r_s' + pL_{r}'
\end{bmatrix}
\begin{bmatrix}
l_{dcs} \\
l_{qcs}'
\end{bmatrix}
\]

Where,
\[
r_s' = \left(\frac{N_s}{N_f}\right)^2 r_s
\]

Energy stored in the coupling field may be written as
\[
W_c' = W_f' = \frac{1}{2}(l_{dcs}')^T (L_{s} - L_{f} I) l_{dcs} + \\
\frac{1}{2}(l_{qcs}')^T (L_{sr}') l_{qcs} + \frac{1}{2}(l_{qcs}')^T (I_{r} - L_{f} I) l_{qcs}'
\]

Where, \(I\): identity matrix

Voltage equations expressed in terms of machine variables referred to the stator windings may be written as
\[
T_c'(\phi, \theta_f) = \frac{\mu}{2} \frac{T_c'(\phi, \theta_f)}{\partial \phi_f}
\]

Since \(L_s\) and \(L_f\) are functions of \(\phi_f\), the above equation for the electromagnetic torque yields.
\[ T_c = \left( \frac{P}{2} \right) \mathcal{L}_{dcr} + \mathcal{C} \left[ \mathbf{L}'_{dcr} \right]_{dcr} \]

\[ = -\frac{P}{2} I_{ms} \left\{ \left[ i_{as} (\theta_{ar} - \frac{1}{2} \theta_{br} - \frac{1}{2} \theta_{cr}) + i_{bs} (\theta_{br} - \frac{1}{2} \theta_{or} - \frac{1}{2} \theta_{cr}) + i_{cs} (\theta_{cr} - \frac{1}{2} \theta_{or} - \frac{1}{2} \theta_{or}) \right] \sin \theta \right\} \frac{\sqrt{3}}{2} \left[ i_{as} (\theta_{ar} - \theta_{cr}) + i_{bs} (\theta_{cr} - \theta_{or}) + i_{cs} (\theta_{or} - \theta_{ar}) \right] \cos \theta \]

The torque and rotor speed are related by

\[ T_c = J \left( \frac{2}{\mu} \right) \rho \omega_r - T_L \]

**Equations of Transformation for Rotor Circuit**

In the analysis of induction machines it is desirable to transform the variables associated with the symmetrical rotor windings to the arbitrary reference frame.

\[ f'_{qcr} = K_r f'_{dcr} \]

\[ \left[ \begin{array}{c} f'_{qcr} \\ f'_{dcr} \end{array} \right] = \left[ \begin{array}{cc} \cos \beta & \cos(\beta - \frac{2\pi}{3}) \\ \cos(\beta + \frac{2\pi}{3}) & \cos(\beta - \frac{2\pi}{3}) \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] \]

where, \( \beta = \theta - \theta_r \) from figure below

![Fig. 2. Axis of 2-pole, 3-phase Symmetrical machine.](image)

\[ \theta_r = \int_0^t \omega_r (t) dt + \theta_r (0) \]
\[
(K_r)^T = \begin{bmatrix}
\cos \beta & \sin \beta & 1 \\
\cos(\beta - \frac{2\pi}{3}) & \sin(\beta - \frac{2\pi}{3}) & 1 \\
\cos(\beta + \frac{2\pi}{3}) & \sin(\beta + \frac{2\pi}{3}) & 1 \\
\end{bmatrix}
\]

“r” subscript indicates the variable, parameters and transformation associated with rotating circuits.

**Voltage Equations in Arbitrary Reference Frame Variables**

For two-pole, 3-phase symmetrical induction,

\[
\begin{align*}
\Gamma_{abcs} &= \bar{r}_{abcs} + p \lambda_{abcs} \\
\lambda_{abcs} &= (\Gamma_{abcs})_{abcs} + (\Gamma_{abcs})_{abcs} \\
\Gamma_{abcr} &= K_r \Gamma_{qrt} \\
\lambda_{abcr} &= K_r \lambda_{qrt}
\end{align*}
\]

Using the above transformation equations, we can transform the voltage equations to an arbitrary reference frame rotating at speed of \( \omega \),

\[
\begin{align*}
\Gamma_{qdis} &= r_{qdis} + \omega \lambda_{qdis} + p \lambda_{qdis} \\
\lambda_{qdis} &= (\Gamma_{qdis})_{qdis} + (\Gamma_{qdis})_{qdis} \\
\Gamma_{qdr} &= K_r \Gamma_{qdr} \\
\lambda_{qdr} &= K_r \lambda_{qdr}
\end{align*}
\]

where,

\[
\begin{bmatrix}
\lambda_{qdis} \\
\lambda_{qdr}
\end{bmatrix} = \begin{bmatrix}
\lambda_{qdis} - \lambda_{qdi} & 0 \\
\lambda_{qdr} - \lambda_{qdr}
\end{bmatrix}
\]

Flux linkage equations in abc reference frame can be transformed to qd axes using \( K_s \) and \( K_r \) transformation matrices.

\[
\begin{bmatrix}
\lambda_{qdis} \\
\lambda_{qdr}
\end{bmatrix} = \begin{bmatrix}
K_s L_s(K_s)^{-1} & K_s L'_s(K_s)^{-1} \\
K_r L'_r(K_r)^{-1} & K_s L'_r(K_r)^{-1}
\end{bmatrix} \begin{bmatrix}
\Gamma_{qdis} \\
\Gamma_{qdr}
\end{bmatrix}
\]

Where

\[
\begin{align*}
K_s L_s(K_s)^{-1} &= \begin{bmatrix}
I_{ls} + M & 0 & 0 \\
0 & I_{ls} + M & 0 \\
0 & 0 & I_{ls} + M
\end{bmatrix} \\
M &= \frac{3}{2} I_{ms} \\
K_r L'_r(K_r)^{-1} &= \begin{bmatrix}
I_{br} + M & 0 & 0 \\
0 & I_{br} + M & 0 \\
0 & 0 & I_{br} + M
\end{bmatrix} \\
M &= \frac{3}{2} I_{ms}
\end{align*}
\]

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\[
\mathbf{K}_s \mathbf{L}'_{qr}(\mathbf{K}_r)^{-1} = \mathbf{K}_r \mathbf{L}'_{qr}'(\mathbf{K}_s)^{-1} = \begin{bmatrix}
M & 0 & 0 \\
0 & M & 0 \\
0 & 0 & M
\end{bmatrix}
\]

Voltage equations written in expanded form can be expressed as

\[
\begin{align*}
V_{qs} &= r_s i_{qs} + \omega \lambda_{ds} + p \lambda_{qs} \\
V_{ds} &= r_s i_{ds} - \omega \lambda_{qs} + p \lambda_{ds} \\
V_{qs} &= r_s i_{qs} + p \lambda_{ds}
\end{align*}
\]

\[
\begin{align*}
V_{qs}' &= r_s' \lambda_{qs}' + (\omega - \omega_r) \lambda_{ds}' + p \lambda_{qs}' \\
V_{ds}' &= r_s' \lambda_{ds}' - (\omega - \omega_r) \lambda_{qs}' + p \lambda_{ds}' \\
V_{qs}' &= r_s' \lambda_{qs}' + p \lambda_{ds}'
\end{align*}
\]

Flux linkage equations are

\[
\begin{align*}
\lambda_{qs} &= I_{q}\mathbf{L}_{qs} + M(i_{qs} + i_{qr}') \\
\lambda_{ds} &= I_{d}\mathbf{L}_{ds} + M(i_{ds} + i_{dr}') \\
\lambda_{qs} &= I_{q}\mathbf{L}_{qs}
\end{align*}
\]

Since machine and power system parameters are nearly always given in ohms or percent or per unit of a base impedance, it is convenient to express the voltage and flux linkage equations in terms of reactances rather than inductances.

Let

\[
\phi = \lambda \omega_b
\]

Then

\[
\begin{align*}
V_{qs} &= r_s i_{qs} + \frac{\omega}{\omega_b} \phi_{qs} + p \phi_{qs} \\
V_{ds} &= r_s i_{ds} - \frac{\omega}{\omega_b} \phi_{qs} + \frac{p}{\omega_b} \phi_{qs} \\
V_{qs} &= r_s i_{qs} + \frac{p}{\omega_b} \phi_{qs}
\end{align*}
\]

\[
\begin{align*}
V_{qs}' &= r_s' \lambda_{qs}' + \frac{(\omega - \omega_r)}{\omega_b} \phi_{ds}' + \frac{p}{\omega_b} \phi_{qs}' \\
V_{ds}' &= r_s' \lambda_{ds}' - \frac{(\omega - \omega_r)}{\omega_b} \phi_{qs}' + \frac{p}{\omega_b} \phi_{ds}' \\
V_{qs}' &= r_s' \lambda_{qs}' + \frac{p}{\omega_b} \phi_{ds}'
\end{align*}
\]

And flux linkages become flux linkages per second with the units of volts

\[
\begin{align*}
\phi_{qs} &= X_{qs} i_{qs} + X_m(i_{qs} + i_{qr}') \\
\phi_{qs} &= X_{qs}' i_{qs}' + X_m(i_{qs} + i_{qr}') \\
\phi_{ds} &= X_{ds} i_{ds} + X_m(i_{ds} + i_{dr}') \\
\phi_{ds} &= X_{ds}' i_{ds}' + X_m(i_{ds} + i_{dr}') \\
\phi_{qs} &= X_{qs} i_{qs}
\end{align*}
\]

\[
\begin{align*}
\phi_{qs}' &= X_{qs}' i_{qs}' + X_m(i_{qs} + i_{qr}') \\
\phi_{ds}' &= X_{ds}' i_{ds}' + X_m(i_{ds} + i_{dr}') \\
\phi_{qs}' &= X_{qs}' i_{qs}'
\end{align*}
\]
q-axis circuit

Equivalent circuits of a 3-phase, symmetrical induction machine with rotating q-d axis at speed of $\omega$.

d-axis circuit

Equivalent circuits of a 3-phase, symmetrical induction machine with rotating q-d axis at speed of $\omega$.

Os-circuit

Equivalent circuits of a 3-phase, symmetrical induction machine with rotating q-d axis at speed of $\omega$.

Or-circuit
Electromagnetic torque in terms of arbitrary reference frame variables may be obtained by substituting the equations of transformation in

\[ T_e = \frac{P}{2} (i_{abc})^T \frac{\partial}{\partial \theta_r} (i'_{sr}) y_{abc} \]

\[ = \frac{P}{2} \left[ (K_T)^{-1} l_{q0s} \right] \frac{\partial}{\partial \theta_r} (i'_{sr}) K_T^{-1} l'_{q0r} \]

After some work, we will have the following:

\[ T_e = \left( \frac{3}{2} \right) \left( \frac{P}{2} \right) M (i_q i_{dr} - i_d i_{qr}) \]

Where, Te is positive for motor action. Other expressions for the electromagnetic torque of an induction machine are

\[ T_e = \left( \frac{3}{2} \right) \left( \frac{P}{2} \right) \left( \lambda_{q0} i_{dr} - \lambda_{d0} i_{qr} \right) \]

\[ T_{cm} = \left( \frac{3}{2} \right) \left( \frac{P}{2} \right) \left( \lambda_{q0} i_{dr} - \lambda_{d0} i_{dr} \right) \]

\[ T_c = \left( \frac{3}{2} \right) \left( \frac{P}{2} \right) \left( \frac{1}{\omega_b} \right) \left( \phi_{q0} i_{dr} - \phi_{d0} i_{qr} \right) \]
The Analysis and Modelling of a Self-excited Induction Generator Driven by a Variable Speed Wind Turbine

1. Introduction

Induction machine is used in a wide variety of applications as a means of converting electric power to mechanical work. The primary advantage of the induction machine is its rugged brushless construction and no need for separate DC field power. These machines are very economical, reliable, and are available in the ranges of fractional horse power (FHP) to multi-megawatt capacity. Also, unlike synchronous machines, induction machines can be operated at variable speeds. For economy and reliability many wind power systems use induction machines, driven by a wind turbine through a gear box, as an electrical generator. The need for gearbox arises from the fact that lower rotational speeds on the wind turbine side should be converted to high rotor speeds, on the electrical generator side, for electrical energy production.

There are two types of induction machine based on the rotor construction namely, squirrel cage type and wound rotor type. Squirrel cage rotor construction is popular because of its ruggedness, lower cost and simplicity of construction and is widely used in stand-alone wind power generation schemes. Wound rotor machine can produce high starting torque and is the preferred choice in grid-connected wind generation scheme. Another advantage with wound rotor is its ability to extract rotor power at the added cost of power electronics in the rotor circuit.

This chapter focuses on the electrical generation part of a wind energy conversion system. After a brief introduction of the induction machine, the electrical generator used in this chapter, a detailed analysis of the induction machine operated in stand-alone mode is presented. As a generator, induction machines have the drawback of requiring reactive power for excitation. This necessitates the use of shunt capacitors in the circuit. The effect of magnetization inductance on self-excitation of the induction generator is discussed. Also, this chapter presents the two existing methods to analyze the process of self-excitation in induction machine and the role of excitation-capacitors in its initiation.

Simulation results of the self-excited induction generator driven by the variable speed wind turbine are presented in the last section of this chapter. The process of voltage build up and the effect of saturation characteristics are also explained in the same section.
2. Induction machine

In the electromagnetic structure of the Induction machine, the stator is made of numerous coils with three groups (phases), and is supplied with three phase current. The three coils are physically spread around the stator periphery (space-phase), and carry currents which are out of time-phase. This combination produces a rotating magnetic field, which is a key feature of the working of the induction machine. Induction machines are asynchronous speed machines, operating below synchronous speed when motoring and above synchronous speed when generating. The presence of negative resistance (i.e., when slip is negative), implies that during the generating mode, power flows from the rotor to the stator in the induction machine.

2.1 Equivalent electrical circuit of induction machine

The theory of operation of induction machine is represented by the per phase equivalent circuit shown in Figure 1 (Krause et al., 1994).

![Per-phase equivalent circuit of the induction machine referred to the stator.](image)

In the above figure, \( R \) and \( X \) refer to the resistance and inductive reactance respectively. Subscripts 1, 2 and \( m \) represent stator, rotor values referred to the stator side and magnetizing components, respectively.

Induction machine needs AC excitation current for its running. The machine is either self-excited or externally excited. Since the excitation current is mainly reactive, a stand-alone system is self-excited by shunt capacitors. In grid-connected operation, it draws excitation power from the network, and its output frequency and voltage values are dictated by the grid. Where the grid capacity of supplying the reactive power is limited, local capacitors can be used to partly supply the needed reactive power (Patel, 1999).

3. Self-Excited Induction Generator (SEIG)

Self-excited induction generator (SEIG) works just like an induction machine in the saturation region except the fact that it has excitation capacitors connected across its stator terminals. These machines are ideal choice for electricity generation in stand-alone variable speed wind energy systems, where reactive power from the grid is not available. The induction generator will self-excite, using the external capacitor, only if the rotor has an adequate remnant magnetic field. In the self-excited mode, the generator output frequency and voltage are affected by the speed, the load, and the capacitance value in farads (Patel, 1999). The steady-state per-phase equivalent circuit of a self-excited induction generator is shown in the Figure 2.
The process of self-excitation in induction machines has been known for many decades (Basset & Potter, 1935). When capacitors are connected across the stator terminals of an induction machine, driven by an external prime mover, voltage will be induced at its terminals. The induced electromotive force (EMF) and current in the stator windings will continue to rise until the steady-state condition is reached, influenced by the magnetic saturation of the machine. At this operating point the voltage and the current will be stabilized at a given peak value and frequency. In order for the self-excitation to occur, for a particular capacitance value there is a corresponding minimum speed (Wagner, 1935). So, in stand-alone mode of operation, it is necessary for the induction generator to be operated in the saturation region. This guarantees one and only one intersection between the magnetization curve and the capacitor reactance line, as well as output voltage stability under load as seen in the Figure 3:

At no-load, the capacitor current $I_c = \frac{V_1}{X_c}$ must be equal to the magnetizing current $I_m = \frac{V_1}{X_m}$. The voltage $V_1$ is a function of $I_m$, linearly rising until the saturation point of the magnetic core is reached. The output frequency of the self-excited generator is $f = \frac{1}{2\pi CX_m}$ and $f = 2\pi f$ where $C$ is self-exciting capacitance.

4. Methods of analysis

There are two fundamental circuit models employed for examining the characteristics of a SEIG. One is the per-phase equivalent circuit which includes the loop-impedance method adopted by (Murthy et al, 1982) and (Malik & Al-Bahrani, 1990), and the nodal admittance method proposed by (Ouazene & Mcpherson, 1983) and (Chan, 1993). This method is
suitable for studying the machine’s steady-state characteristics. The other method is the dq-axis model based on the generalized machine theory proposed by (Elder et al., 1984) and (Grantham et al., 1989), and is employed to analyze the machine’s transient state as well as steady-state.

4.1 Steady-state model

Steady-state analysis of induction generators is of interest both from the design and operational points of view. By knowing the parameters of the machine, it is possible to determine the performance of the machine at a given speed, capacitance and load conditions. Loop impedance and nodal admittance methods used for the analysis of SEIG are both based on per-phase steady-state equivalent circuit of the induction machine (Figure 4), modified for the self-excitation case. They make use of the principle of conservation of active and reactive powers, by writing loop equations (Murthy et al., 1982), (Malik & Al-Bahrani, 1990), (Al-Jabri & Alolah, 1990) or nodal equations (Ouazene & Mcpherson, 1983), (Chan, 1993), for the equivalent circuit. These methods are very effective in calculating the minimum value of capacitance needed for guaranteeing self-excitation of the induction generator. For stable operation, excitation capacitance must be slightly higher than the minimum value. Also there is a speed threshold, below which no excitation is possible, called as the cutoff speed of the machine. In the following paragraph, of loop impedance method is given for better understanding.

The per-unit per-phase steady-state circuit of a self-excited induction generator under RL load is shown in Figure 4 (Murthy et al., 1982), (Ouazene & Mcpherson, 1983).

Fig. 4. Equivalent circuit of self-excited induction generator with R-L Load.

Where:

- \( R_s, R_r, R \) : p.u. per-phase stator, rotor (referred to stator) and load resistance respectively.
- \( X_{ls}, X_{lr}, X, X_m \) : p.u. per-phase stator leakage, rotor leakage (referred to stator), load and magnetizing reactances (at base frequency), respectively.
- \( X_{max} \) : p.u. maximum saturated magnetizing reactance.
- \( C \) : per-phase terminal excitation capacitance.
- \( X_C \) : p.u. per-phase capacitive reactance (at base frequency) of the terminal excitation capacitor.
- \( f, v \) : p.u. frequency and speed, respectively.
- \( N \) : base speed in rev/ min
- \( Z_b \) : per-phase base impedance
- \( f_b \) : base frequency
- \( V_g, V_o \) : per-phase air gap and output voltages, respectively.

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In the analysis of SEIG the following assumptions were made (Murthy et al., 1982):

1. Only the magnetizing reactance $X_m$ is assumed to be affected by magnetic saturation, and all other parameters of the equivalent circuit are assumed to be constant. Self-excitation results in the saturation of the main flux and the value of $X_m$ reflects the magnitude of the main flux. Leakage flux passes mainly in the air, and thus these fluxes are not affected to any large extent by the saturation of the main flux.

2. Stator and rotor leakage reactance, in per-unit are taken to be equal. This assumption is normally valid in induction machine analysis.

3. Core loss in the machine is neglected.

For the circuit shown in Figure 4, the loop equation for the current can be written as:

$$I = Z = 0 \quad (1)$$

Where $Z$ is the net loop impedance given by

$$Z = \left( \frac{R_s}{f - v} + jX_m \right) + \frac{R_s}{f} + jX_s + \left( \frac{R_s}{f^2} \right) \left( \frac{R_s}{f} + jX_s \right) \quad (2)$$

Since at steady-state excitation $I \neq 0$, it follows from (equation 1) that $Z = 0$, which implies that both the real and imaginary parts of $Z$ are zeros. These two equations can be solved simultaneously for any two unknowns (usually voltage and frequency). For successful voltage-buildup, the load-capacitance combination and the rotor speed should result in a value such that $X_m = X_{m_{max}}$, which yields the minimum value of excitation capacitance below which the SEIG fails to self-excite.

4.2 Steady-state and transient model (abc-dq0 transformation)

The process of self-excitation is a transient phenomenon and is better understood if analyzed using a transient model. To arrive at transient model of an induction generator, abc-dq0 transformation is used.

4.2.1 abc-dq0 transformation

The abc-dq0 transformation transfers an abc (in any reference frame) system to a rotating dq0 system. (Krause et al., 1994) noted that, all time varying inductances can be eliminated by referring the stator and rotor variables to a frame of reference rotating at any angular velocity or remaining stationary. All transformations are then obtained by assigning the appropriate speed of rotation to this (arbitrary) reference frame. Also, if the system is balanced the zero component will be equal to zero (Krause et al., 1994).

A change of variables which formulates a transformation of the 3-phase variables of stationary circuit elements to the arbitrary reference frame may be expressed as (Krause et al., 1994):

$$F_{qdos} = K_{sabc}$$

Where:

$$F_{qdos} = \begin{bmatrix} f_{qs} \\ f_{ds} \\ f_{os} \end{bmatrix}, K_{sabc} = \frac{2}{3} \begin{bmatrix} \cos \theta & \cos \left( \theta - \frac{2\pi}{3} \right) & \cos \left( \theta + \frac{2\pi}{3} \right) \\ \sin \theta & \sin \left( \theta - \frac{2\pi}{3} \right) & \sin \left( \theta + \frac{2\pi}{3} \right) \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \cdot \theta = \int_0^\infty \omega(\xi) d\xi + \theta(0)$$
is the dummy variable of integration

For the inverse transformation:

\[
(K)_{-1} = \begin{bmatrix}
\cos \beta & \sin \beta & 1 \\
\cos \left(\beta - \frac{2\pi}{3}\right) & \sin \left(\beta - \frac{2\pi}{3}\right) & 1 \\
\cos \left(\beta + \frac{2\pi}{3}\right) & \sin \left(\beta + \frac{2\pi}{3}\right) & 1 \\
\end{bmatrix}
\]

In (equation 3), \( f \) can represent voltage, current, flux linkage, or electric charge. The subscript \( s \) indicates the variables, parameters and transformation associated with stationary circuits. This above transformation could also be used to transform the time-varying rotor windings of the induction machine. It is convenient to visualize the transformation equations as trigonometric relationships between variables as shown in Figure 5.

Fig. 5. Transformation for stationary circuits portrayed by trigonometric relationships.

The equations of transformation may be thought of as if the \( f_{qs} \) and \( f_{ds} \) variables are directed along axes orthogonal to each other and rotating at an angular velocity of \( \omega \), where upon \( f_{as} \), \( f_{bs} \), and \( f_{cs} \) (instantaneous quantities which may be any function of time), considered as variables directed along stationary paths each displaced by 120°. Although the waveforms of the \( qs \) and \( ds \) voltages, currents and flux linkages, and electric charges are dependent upon the angular velocity of the frame of reference, the waveform of the total power is same regardless of the reference frame in which it is evaluated (Krause et al., 1994).

4.2.2 Voltage equations in arbitrary reference-frame variables

The winding arrangement for a 2-pole, 3-phase, wye-connected, symmetrical induction machine is shown in Figure 6.

The stator windings are identical sinusoidally distributed windings, displaced 120°, with \( N_s \) equivalent turns and resistance \( r_s \). The rotor consists of three identical sinusoidally distributed windings, with \( N_r \) equivalent turns and resistance \( r_r \). Note that positive a, b, c sequence is used in both in Figures 5 and 6.

The voltage equations in machine variables can be expressed as (Krause et al., 1994):
Fig. 6. Two-pole, 3-phase, wye connected symmetrical induction machine.

\[ V_{abcs} = r_{s}i_{abcs} + p\lambda_{abcs} \]  \hspace{1cm} (4)

\[ V_{abc} = r_{l}i_{abc} + p\lambda_{abc} \]  \hspace{1cm} (5)

Where:

Subscript \( s \) denotes parameters and variables associated with the stator.

Subscript \( r \) denotes parameters and variables associated with the rotor.

\( V_{abcs}, V_{abc} \) are phase voltages.

\( i_{abcs}, i_{abc} \) are phase currents.

\( \lambda_{abcs}, \lambda_{abc} \) are the flux linkages and \( p = \frac{d}{dt} \).

By using the abc-dq0 transformation and expressing flux linkages as product of currents and winding inductances, we obtain the following expressions for voltage in arbitrary reference frame (Krause et al., 1994):

\[ V_{qdos} = r_{s}i_{qdos} + \omega_{f}\lambda_{qdos} + p\lambda_{qdos} \]  \hspace{1cm} (6)

\[ V'_{qdor} = r_{r}i'_{qdor} + (\omega_{f} - \omega_{r})\lambda_{qdor} + p\lambda'_{qdor} \]  \hspace{1cm} (7)

Where:

\( \omega_{f} \) is the electrical angular velocity of the arbitrary frame.

\( \omega_{r} \) is the electrical angular velocity of the rotor.

\( (\lambda_{qdos})^{T} = [\lambda_{ds} \hspace{0.5cm} \lambda_{qs} \hspace{0.5cm} 0] \); \( (\lambda'_{qdor})^{T} = [\lambda'_{dr} \hspace{0.5cm} \lambda'_{qr} \hspace{0.5cm} 0] \)

\( ' \) denotes rotor values referred to the stator side.

Using the relations between the flux linkages and currents in the arbitrary reference frame and substituting them in (equations 6) & (equations 7), the voltage and flux equations are expressed as follows:

\[ V_{q} = r_{qs}i_{qs} + \omega_{f}\lambda_{qs} + p\lambda_{qs} \]  \hspace{1cm} (8)

\[ V_{ds} = r_{ds}i_{ds} + \omega_{f}\lambda_{ds} + p\lambda_{ds} \]  \hspace{1cm} (9)
\[
V'_{qr} = r' i'_{qr} \left( \omega \cdot \omega \right) V'_{dr} + p V'_{qr} \tag{10}
\]
\[
V'_{dr} = r' i'_{dr} \cdot \left( \omega \cdot \omega \right) V'_{qr} + p V'_{dr} \tag{11}
\]
\[
\lambda_{ds} = L_{ds} i_{ds} + L_{m} (i_{ds} + i'_{dr}) \tag{12}
\]
\[
\lambda_{dr} = L_{dr} i_{dr} + L_{m} (i_{dr} + i'_{dr}) \tag{13}
\]
\[
\lambda'_{ds} = L_{ds}' i_{ds} + L_{m} (i_{ds} + i'_{dr}) \tag{14}
\]
\[
\lambda'_{dr} = L_{dr}' i_{dr} + L_{m} (i_{dr} + i'_{dr}) \tag{15}
\]

Where:
- \( L_{ls} \) and \( L_{ms} \) are leakage and magnetizing inductances of the stator respectively.
- \( L_{lr} \) and \( L_{mr} \) are leakage and magnetizing inductances of the rotor respectively.
- Magnetizing inductance, \( L_{m} = \frac{3}{2} L_{ms} \)

The voltage and flux linkage equations suggest the following equivalent circuits for the induction machine:

![Arbitrary reference-frame equivalent circuits for a 3-phase, symmetrical induction machine](www.ravivarmans.com)

### 4.2.2.1 Torque equations

The expression for electromagnetic torque, positive for motor operation and negative for generator operation, in terms of the arbitrary reference variables can be expressed as (Krause et al., 1994):

For motor action, \( T_e = \left( \frac{2}{p} \right) L_m (i_{ds}' i_{dr} - i_{ds} i_{qr}') \) \tag{16}

For generator action, \( T_e = \left( \frac{2}{p} \right) L_m (i_{ds} i_{qr}' - i_{ds}' i_{dr}) \) \tag{17}

The torque and speed are related by the following expressions:

For the motor operation, \( T_{e,\text{motor}} = J \left( \frac{2}{p} \right) p \omega + T_D \) \tag{18}

For the generator operation, \( T_D = J \left( \frac{2}{p} \right) p \omega + T_{\text{e,gen}} \) \tag{19}
Where:
P: Number of poles.
J: Inertia of the rotor in (Kg m²).
T₀: Drive torque in (Nm).

4.2.3 Stationary reference frame
Although the behavior of the induction machine may be described by any frame of reference, there are three which are commonly used (Krause et al., 1994). The voltage equations for each of these reference frames can be obtained from the voltage equations in the arbitrary reference frame by assigning the appropriate speed to \(\omega\). That is, for the stationary reference frame, \(\omega = 0\), for the rotor reference frame, \(\omega = \omega_r\), and for the synchronous reference frame, \(\omega = \omega_e\).

Generally, the conditions of operation will determine the most convenient reference frame for analysis and/or simulation purposes. The stator reference frame is used when the stator voltages are unbalanced or discontinuous and the rotor applied voltages are balanced or zero. The rotor reference frame is used when the rotor voltages are unbalanced or discontinuous and the stator applied voltages are balanced. The stationary frame is used when all (stator and rotor) voltages are balanced and continuous. In this thesis, the stationary reference frame (\(\omega = 0\)) is used for simulating the model of the self-excited induction generator (SEIG).

In all asynchronously rotating reference frames \(\omega(\neq \omega_r)\) with \(\omega(0)=0\) (see (equation 3)), the phasor representing phase variables (with subscript as) is equal to phasor representing qs variables. In other terms, for the rotor reference frame and the stationary frame, \(f_{as} = f_{qs}\), \(f_{bs} = f_{as}\) and \(f_{cs} = f_{as}\).

4.2.4 SEIG model
As discussed above, the dq model of the SEIG in the stationary reference frame is obtained by substituting \(\omega = 0\) in the arbitrary reference frame equivalent of the induction machine shown in Figure 7. Figure 8 shows a complete dq-axis model, of the SEIG with load, in the stationary reference frame. Capacitor is connected at the stator terminals for the self-excitation. For convenience, all values are assumed to be referred to the stator side and here after "'" is neglected while expressing rotor parameters referred to the stator (Seyoum et al., 2003).

![Fig. 8. dq model of SEIG in stationary reference frame (All values referred to stator)](www.ravivarmans.com)
For no-load condition, rearranging the terms after writing loop equations for Figure 8, we obtain the following voltage equations expressed in the form of a matrix (Krause et al., 1994), (Seyoum et al., 2003):

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= 
\begin{bmatrix}
I_r + pL_s + 1/pC \\
0 \\
a_0 I_m
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
V_{c_q} \\
V_{c_d} \\
K_s
\end{bmatrix}
\]

(20)

Where:
- \(K_s\) and \(K_q\) are constants representing initial induced voltages along the d-axis and q-axis respectively, due to the remaining magnetic flux in the core.
- \(V_{c_q}\) and \(V_{c_d}\) are initial voltages in the capacitors.

\[L_s = L_{ds} + L_{ms}\]
and
\[L_r = L_{dr} + L_{mr}\]

The above equations can further be simplified in the following manner using (equations 8-15):

In the stationary reference frame (equations 8) can be written as,

\[V_{qs} = -V_{cq} + 0 \times \lambda_{ds} + p\lambda_{qs}\]

Substituting (equation 12) in (equation 21), will result

\[0 = V_{cq} + r_s i_{qs} + L_s i_{qs} + L_m i_{qr}\]

(22)

Solving for \(i_{qr}\) by substituting (equation 14) and (equation 15) in (equation 10) yields:

\[i_{qr} = \frac{1}{L_s}(V_{cq} - \omega_r L_s i_{qs} + \omega_r L_m i_{ds} - L_m K_q - L_r V_{cq})\]

(23)

Substituting (equation 23) in (equation 22) results in the final expression for \(i_{qs}\) as:

\[p i_{qs} = \frac{1}{L_s}(-L_s^2 \omega_r i_{qs} + \omega_r L_m i_{ds} - L_m K_q - L_r V_{cq})\]

(24)

Where:

\[L = L_{ds} \cdot \frac{1}{L_{ds}}\]

Similarly, the expressions for other current components are obtained and the SEIG can be represented in a matrix form as:

\[pi = Ai + B\]

Where:

\[A = \frac{L}{L_s} \begin{bmatrix}
-L_s r_s & \omega_r L_s & L_m r_s & -L_m \omega_r L_s \\
L_s^2 r_s & -L_s r_s & L_s \omega_r L_s & L_m r_s \\
-L_s^2 \omega_r L_m & L_s r_s & -L_s^2 \omega_r L_m & L_s \omega_r L_m \\
-L_s \omega_r L_m & L_s^2 \omega_r L_m & L_s r_s & -L_s \omega_r L_m
\end{bmatrix}\]

\[B = \frac{1}{L_s} \begin{bmatrix}
L_m K_q - L_r V_{cq} \\
L_m K_q - L_r V_{cq} \\
L_m K_q - L_r V_{cq} \\
L_m K_q - L_r V_{cq}
\end{bmatrix}\]
\[ \oint = \frac{\dot{q}_s}{\dot{q}_f} \]  
\[ \int \frac{1}{\dot{q}_s} \, dq + \frac{v_{cq}}{\dot{q}_s} \, dt = 0 \; ; \; \frac{1}{\dot{q}_s} \, dq + \frac{v_{cq}}{\dot{q}_s} \, dt = 0 \]

Any combination of R, L and C can be added in parallel with the self-excitation capacitance to act as load. For example, if resistance R is added in parallel with the self-excitation capacitance, then the term 1/pC in (equation 20) becomes R/(1+RpC). The load can be connected across the capacitors, once the voltage reaches a steady-state value (Grantham et al., 1989), (Seyoum et al., 2003).

The type of load connected to the SEIG is a real concern for voltage regulation. In general, large resistive and inductive loads can vary the terminal voltage over a wide range. For example, the effect of an inductive load in parallel with the excitation capacitor will reduce the resulting effective load impedance (Z_{eff}) (Simoes & Farret, 2004).

\[ Z_{eff} = R + j \left( \omega L - \frac{1}{\omega C} \right) \]  \hspace{1cm} (26)

This change in the effective self-excitation increases the slope of the straight line of the capacitive reactance (Figure 3), reducing the terminal voltage. This phenomenon is more pronounced when the load becomes highly inductive.
Self-Excited Induction Generators

4.1 Introduction

By self-excited induction generators (SEIGs), we mean cage rotor induction machines with shunt (and series) capacitors connected at their terminals for self-excitation.

The shunt capacitors may be constant or may be varied through power electronics (or step-wise). SEIGs may be built with single-phase or three-phase output and may supply alternating current (AC) loads or AC rectified (direct current [DC]) autonomous loads. We also include here SEIGs connected to the power grid through soft-starters or resistors and having capacitors at their terminals for power factor compensation (or voltage stabilization).

Note that power electronics controlled cage rotor induction generators (IGs) for constant voltage and frequency output at variable speed, for autonomous and power grid operation, will be treated in Chapter 5.

This chapter will introduce the main schemes for SEIGs and their steady-state and transient performance, with sample results for applications such as wind machines, small hydrogenerators, or generator
sets. Both power grid and stand-alone operation and three-phase and single-phase output SEIGs are treated in this chapter.

4.2 The Cage Rotor Induction Machine Principle

The cage rotor induction machine is the most built and most used electric machine, mainly as a motor, but, recently, as a generator, too.

The cage rotor induction machine contains cylindrical stator and rotor cores with uniform slots separated by a small airgap (0.3 to 2 mm in general).

The stator slots host a three-phase or a two-phase AC winding meant to produce a traveling magnetomotive force (mmf). The windings are similar to those described for synchronous generators (SGs) in Chapter 4 of Synchronous Generators or for wound rotor induction generators (WRIGs) in Chapter 3 of this book. This traveling mmf produces a traveling flux density in the airgap, $B_{x10}$:

$$B_{x10} = \frac{\mu_0 F_{10}}{g} \cos(\omega t - p_1 \theta_r)$$  \hspace{1cm} (4.1)

$$g = \text{airgap}$$

$$F_{10} = \frac{3\sqrt{2} I_{10} W K_{WL}}{\pi p_1} \quad \text{(for three phases)}$$  \hspace{1cm} (4.2)

where

$\theta_r$ is the rotor position

$p_1$ equals the pole pairs

The cage rotor contains aluminum (or copper, or brass) bars in slots. They are short-circuited by end-rings with resistances that are smaller than those of bars (Figure 4.1).

The angular speed of the traveling fields is obtained for the following:

$$\omega t - p_1 \theta_r = \text{const.}$$  \hspace{1cm} (4.3)

That is, for

$$\frac{d\theta_r}{dt} = \frac{\omega}{p_1}; \quad \eta_1 = \frac{f_1}{p_1}$$  \hspace{1cm} (4.4)

![End rings Bars embedded in slots](FIGURE 4.1 The cage rotor.)
The speed \( n_1 \) (in revolutions per second \( r/sec \)) is the so-called ideal no-load or synchronous speed and is proportional to stator frequency and inversely proportional to the number of pole pairs \( p_1 \).

The traveling field in the airgap induces electromagnetic fields (emfs) in the rotor that rotate at speed \( n \), at frequency \( f_2 \):

\[
f_2 = \left( \frac{n_1 - n}{n_1} \right) f_1 = S f_1
\]

\[
S = \frac{n_1 - n}{n_1}
\]  

(4.5)

As expected, the emfs induced in the short-circuited rotor bars produce in them AC currents at slip frequency \( f_2 = S f_1 \).

Let us now assume that the symmetric rotor cage, which has the property to adapt to almost any number of pole pairs in the stator, may be replaced by an equivalent (fictitious) three-phase symmetric three-phase winding (as in WRIGs) that is short-circuited. The traveling airgap field produces symmetric emfs in the fictitious three-phase rotor with frequency that is \( S f_1 \) and with amplitude that is also proportional to slip \( S \):

\[
E_z = SE_1 = S\omega_1 L_m I_m
\]  

(4.6)

where \( L_m \) is the magnetization inductance.

\( E_1 \) is the stator phase self-induced emf, generally produced by both stator and rotor currents, or by the so-called magnetization current \( I_m \) (\( I_m = I_1 + I_2 \)).

The rotor phases may be represented by a leakage inductance \( L_{2l} \) and a resistance \( R_2 \). Consequently, the rotor current \( I_2 \) is as follows:

\[
I_2 = \frac{SE_1}{\sqrt{(R_2)^2 + (S\omega_1 L_{2l})^2}}
\]  

(4.7)

The rotor currents interact with the airgap field to produce tangential forces — torque. In Equation 4.6 and Equation 4.7, the rotor winding is reduced to the stator winding based on energy (and loss) equivalence.

Noticing that the stator phases are also characterized by a resistance \( R_1 \) and a leakage inductance \( L_{1p} \), the stator and rotor equations may be written, for steady state, in complex numbers, as for a transformer but with different frequencies in the primary and secondary. Let us consider the generator association of signs for the stator:

\[
L_1 (R_1 + j\omega_1 L_{1p}) + V_1 = E_1
\]

\[
L_2 (R_2 + jS\omega_1 L_{2l}) = SE_1
\]

\[
E_1 = -j\omega_1 L_m (I_1 + I_2)
\]  

(4.8)

Dividing the second expression in Equation 4.8 by \( S \) yields the following:

\[
I_2 \left( R_2 + \frac{R_1 (1-S)}{S} + j\omega_1 L_{2l} \right) = E_1
\]  

(4.9)
This way, in fact, the frequency of rotor variables becomes \( \omega_1 \), and it refers to a machine at standstill, but with an additional (fictitious) rotor resistance \( R_2(1 - S)/S \). The power dissipated in this resistance equals the mechanical power in the real machine (minus the mechanical losses):

\[
T_e = 2\pi n_1 (1 - S) = 3I_2^2R_2 \frac{(1 - S)}{S} \tag{4.10}
\]

Finally,

\[
T_e = \frac{3p}{\omega_1} I_2^2 \frac{R_2}{S} = 3 \frac{p}{\omega_1} P_{elm} \tag{4.11}
\]

\( P_{elm} \) is the so-called electromagnetic power: the total active power that crosses the airgap. Equation 4.8 and Equation 4.9 lead to the standard equivalent circuit of the induction machine (IM) with cage rotor (Figure 4.2).

The core loss resistance \( R_m \) is added to account for fundamental core losses located in the stator, as \( S \times 1 \), in general. \( R_m \) is determined by tests or calculated in the design stage. As can be seen from Equation 4.11, the electromagnetic power \( P_{elm} \) is positive (motoring) for \( S > 0 \) and negative (generating) for \( S < 0 \). For details on parameter expressions, various losses, parasitic torques, design, and so forth, of cage rotor IMs, see Reference [1].

As seen from Figure 4.2, the equivalent (total) reactance of the IM is always inductive, irrespective of slip sign (motor or generator), while the equivalent resistance changes sign for generating. So, the IM takes the reactive power to get magnetized either from the power grid to which it is connected or from a fixed (or controlled) capacitor at terminals. Note that when a full power static converter is placed between the IG and the load (or power grid), the IG is again self-excited by the capacitors in the converter’s DC link or from the power grid (if a direct AC–AC converter is used).

As the operation of an IM at the power grid is straightforward \((S < 0, \omega_0 > \omega_0)\) the capacitor-excited induction generator will be treated here first in detail.

### 4.3 Self-Excitation: A Qualitative View

The IG with capacitor excitation is driven by a prime mover with the main power switch open (Figure 4.3a). As the speed increases, due to prime-mover torque, eventually, the no-load terminal voltage increases and settles to a certain value, depending on machine speed, capacitance, and machine parameters.
FIGURE 4.3  Self-excitation on self-excited induction generator (SEIG): (a) the general scheme, (b) oversimplified equivalent circuit, and (c) quasi-steady-state self-excitation characteristics.

The equivalent circuit (Figure 4.2) is further simplified by neglecting the stator resistance and leakage inductance and by considering zero slip ($S = 0$: open rotor circuit) for no-load conditions (Figure 4.3b). $E_{rem}$ represents the no-load initial stator voltage (before self-excitation), at frequency $\omega_{10} = \omega_r$, produced by the remnant flux density in the rotor left there from previous operation events.

To initiate the self-excitation process, $E_{rem}$ has to be nonzero.

The magnetization curve of the IG, obtained from typical motor no-load tests, $E_i(I_m)$, has to advance to the nonlinear (saturation) zone in order to firmly intersect the capacitor straight-line voltage characteristic (Figure 4.3c) and, thus, produce the no-load voltage $E_i$. The process of self-excitation of IG has been known for a long time [2].

The increasing of the terminal voltage from $V_{rem}$ to $V_{10}$ unfolds slowly in time (seconds), and Figure 4.3c presents it as a step-wise quasi-steady-state process. It is a qualitative representation only. Once the SEIG
is self-excited, the load is connected. If the load is purely resistive, the terminal voltage decreases and so does (slightly) the frequency \( \omega_t \) for constant (regulated) prime-mover speed \( \omega_r \).

With \( \omega_t < \omega_r \), the SEIG delivers power to the load for negative slip \( S < 0 \):

\[
f_t = \frac{n p_t}{1 + |S|}; \quad S < 0
\]  

(4.12)

The computation of terminal voltage \( V_t \), frequency \( f_t \), stator current \( I_t \), delivered active and reactive power (efficiency) for given load (speed \( n \)), capacitor \( C \), and machine parameters \( R_1, R_2, L_{1p}, L_{2p}, L_m(I_m) \) represents, in fact, the process of obtaining the steady-state performance.

The nonlinear function \( L_m(I_m) \) — magnetization curve — and the variation of frequency \( f_t \) with load, at constant speed \( n \), make the process mathematically intricate.

### 4.4 Steady-State Performance of Three-Phase SEIGs

Various analytical (and numerical) methods to calculate the steady-state performance of SEIGs were proposed. They seem, however, to fall into two main categories:

- Loop impedance models [3]
- Nodal admittance models [4]

Both models are based on the SEIG equivalent circuit (Figure 4.2) expressed in per unit (P.U.) form for frequency \( f \) (P.U.) and speed \( U \) (in P.U.) as follows:

\[
f = \frac{f_t}{f_{b}^*};
\]

\[
U = np_t f_{b}
\]  

(4.13)

The base frequency for which all reactances \( X_{1p}, X_{2p}, X_m(I_m) \) are calculated is \( f_{b}^* \).

With an \( R_D, L_D, C_L \) load, the equivalent circuit in Figure 4.2 with speed and frequency in P.U. terms becomes as shown in Figure 4.4.

![Self-excited induction generator (SEIG) equivalent circuit in per unit (P.U.) frequency \( f \) and speed \( U \).](www.ravivarman.com)
The presence of frequency $f$ in the load, the dependence of core loss resistance $R_m$ of frequency $f$, and the nonlinear dependence on $X_m$ of $I_m$ makes the solving of the equivalent circuit difficult. The SEIG plus load show zero total impedance:

$$R_e (IG + load) = 0$$

$$X_e (IG + excitation _ capacitor + load) = 0$$

(4.14)

for self-excitation, under load.

To solve it simply, the problem is reduced to two unknowns: $f$ (frequency) and $X_m$ for given excitation capacitor, IG ($R_1, R_2, X_{1p}, X_{2p}, X_m[I_m]$), load ($R_L, X_L, X_C$), and speed $U$.

Two main impedance approaches to solve Equation 4.14 were developed:

- High-order polynomial equation (in $f$) approaches [5]
- Optimization approaches [6, 7]

The high-order polynomial and the optimization method solutions obscure the intuitive understanding of performance sensitivity to various parameters, but they constitute mighty computerized tools.

In References [8, 10], admittance models that led to a quadratic equation for slip $f - U = S$ (instead of $f$) for given frequency $f$, were introduced for balanced resistive loads (RLs) without additional simplifying assumptions. A simple iterative method is used to adjust frequency until the desired speed is obtained.

For the sake of simplicity, the admittance model will be used in what follows.

### 4.4.1 Second-Order Slip Equation Methods

The standard equivalent circuit of Figure 4.4 may be changed by lumping together the IG stator ($R_1, jfX_{1p}$), the excitation capacitor reactance ($-jX_c/f$), and the load ($R_L, jfX_L, -jX_C/f$) into an equivalent series circuit ($R_{1e}, jfX_{1e}$) (Figure 4.5). For self-excitation, $X_{1e} \leq 0$:

$$R_{1e} + jf\omega L_{1e} = R_1 + jf\omega L_{1l} + \frac{-jX_c}{f} \left( R_L + jfX_L - j\frac{X_{cl}}{f} \right)$$

$$R_{1e} + jf\omega L_{1e} = R_L + jf\left( fX_L - \frac{X_C}{f} - \frac{X_{cl}}{f} \right)$$

(4.15)

![FIGURE 4.5 The nodal equivalent circuit of a self-excited induction generator (SEIG).](www.ravivarmans.com)
In general, both \( R_{ll} \) and \( X_{ll} \) are dependent on frequency \( f \) (P.U.), though they get simplified forms if only a resistive or an \( R_{ij}, X_{ij} \) (an induction motor) is considered.

For self-excitation, the summation of currents in the node should be zero (with \( E_1 \neq 0 \)):

\[
-I_1 + I_m - I_2 = 0
\]

or

\[
fE_i \left( \frac{1}{R_{ll} + jfX_{ll}} + \frac{S}{R_2 + jfSX_{2l} + \frac{1}{jfX_m}} \right) = 0
\]  

(4.16)

The real and imaginary parts in Equation 4.17 have to be zero for self-excitation (it is, in fact, an energy balance condition):

\[
\frac{R_{ll}}{R_{ll}^2 + f^2X_{ll}^2} + \frac{SR_2}{R_2^2 + S^2f^2X_{2l}^2} = 0
\]  

(4.18)

\[
\frac{1}{fX_m} - \frac{fX_{ll}}{R_{ll}^2 + f^2X_{ll}^2} + \frac{SfX_{2l}}{R_2^2 + S^2f^2X_{2l}^2} = 0
\]  

(4.19)

For given frequency \( f \) (P.U.), Equation 4.18 remains (for given excitation capacitors, IG parameters, and load), with only one unknown, the slip \( S \):

\[
aS^2 + bS + c = 0
\]  

(4.20a)

with

\[
a = f^2X_{2l}^2R_{ll},
\]

\[
b = R_2 \left( R_{ll}^2 + f^2X_{ll}^2 \right)
\]

\[
c = R_{ll}R_2^2
\]  

(4.20b)

Equation 4.20 has two solutions, but only the smaller one (in amplitude) is really useful. For the larger one, most of the power is consumed into the rotor resistance:

\[
S_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]  

(4.21)

If complex solutions of \( S \) are obtained, it means that self-excitation is impossible.
With slip \( S \), found from Equation 4.21, for given \( f \), the corresponding P.U. speed \( U = f - S \) is determined. With \( S = S_1 \), from Equation 4.19, the magnetization reactance \( X_m \) is calculated as follows:

\[
X_m = f^{-1} \left[ \frac{fX_{1l}}{R_{1l}^2 + f^2X_{1l}^2} - \frac{fSfX_{2l}}{R_S^2 + S^2f^2X_{2l}^2} \right] < X_{max}
\]  

(4.22)

\( X_{max} \) is the maximum (unsaturated) value of the magnetization reactance (at base frequency \( f_{ib} \)). With \( X_m > X_{max} \), self-excitation is again impossible.

Further on, from no-load motor testing, or from design calculations, the \( E_i(I_m) \) or \( X_m(I_m) = E_i/I_m \) characteristic will be determined (Figure 4.6).

\( E_i(X_m) \) from Figure 4.6 may be curve fitted by mathematical approximations such as the following [11]:

\[
E_i = \omega_b K_1 I_m ; \quad I_m < I_0
\]

\[
E_i = \omega_b \left[ K_1 I_m + \frac{K_2}{d} \tan^{-1}(d(I_m - I_0)) \right]
\]

(4.23)

for \( I_m \leq I_0 \).

The coefficients \( K_1, K_2, d \) are calculated to preserve continuity at \( I_m = I_0 \) in \( E_i \) and in \( dE_i/dI_m \), and they reasonably approximate the entire curve. This particular approximation has a steady decrease in the derivative, and its inverse is readily available:

\[
X_m = X_{max} = K_2/\omega_b \quad \text{for } I_m < I_0
\]

(4.24)

\[
E_i = X_m \left[ I_0 + \frac{1}{d} \tan \left( \frac{E_i(1-X_{max}/X_m)}{dK_2\omega_b} \right) \right] ; \quad I_m > I_0
\]

(4.25)

Though Equation 4.25 is a transcendent equation, its numerical solution in \( E_i \), for the now calculated \( X_m \) (Equation 4.22), is rather straightforward.

**FIGURE 4.6** Magnetization curve at base frequency \( f_{ib} \).
Once $E_1$ is known, the equivalent circuit in Figure 4.5 produces all required variables:

$$I_2 = \frac{-fE_1}{R_s + jfX_{2l}}; \quad I_m = \frac{E_1}{jX_m}$$

$$I_1 = \frac{-fE_1}{R_{1l} + jfX_{1l}}; \quad X_{1l} < 0;$$

$$-V_1 = fE_1 + (R_i + jfX_i)I_1 \quad (4.26)$$

$$I_C = -V_i \frac{1}{jX_c}; \quad X_c = \frac{1}{\alpha C}$$

$$I_s = I_1 + I_C$$

$$I_m = I_1 + I_2$$

Let us now draw a general phasor diagram for a typical $R_i, L_i$ load when the load current $I_s$ is lagging behind the terminal voltage $V_1$. Also, notice in Equation 4.26 that $I_1$ is leading $fE_1$, because $V_{1l} < 0$ to fulfill the self-excitation conditions.

The phasor diagram starts with $fE_1$ in the real axis and $I_1$ leading it (Figure 4.7). Then, from Equation 4.26 (the third expression), $V_1$ is constructed. Also, from Equation 4.26 (the first expression), for $S < 0$, $I_2$ is ahead of $E_1$. For resistive-inductive load, the capacitor current is in a leading position with respect to terminal voltage.

The whole computation process described so far may be computerized, and, for given speed $U$ (P.U.), the initial value of $f$ may be taken as $f(1) = U$. After one computation cycle, the slip $S(1)$ is calculated, and the new value $f(2)$ is $f(2) = \nu + S(1)$. The whole iterative process continues until the frequency error between two successive computation cycles is smaller than a desired value.

It was demonstrated [9] that less than ten cycles are required, even if the core loss resistance ($R_m$) would be included. It was also shown that core losses do not modify the machine capability, except for the situation around maximum power delivery.

Once $fE_1$ is known, power core losses $p_{worn}$ may be calculated as follows:

$$p_{worn} = \frac{3(fE_1)^2}{R_m} \quad (4.27)$$

So, the efficiency on SEIG is

$$\eta = \frac{3V_i I_L \cos \phi}{3V_i I_L \cos \phi + 3R_1 I_1^2 + 3R_2 I_2^2 + p_{worn} + p_{mec} + p_{stray} + p_{cap}} \quad (4.28)$$

In Equation 4.28, $p_{mec}$ is the mechanical loss, $p_{stray}$ is the IG stray load loss (Reference [1], Chapter 3), and $p_{cap}$ is the excitation capacitor loss.

![FIGURE 4.7 The phasor diagram.](www.ravivarmans.com)
FIGURE 4.8 Voltage vs. load current of self-excited induction generator (SEIG) with shunt self-excitation.

Typical load voltage $V_l$ vs. load current $I_l$ for unity power factor load ($\varphi = 0$), for a 20 kW, four-pole, 50 Hz, 380 V (Y), 5.4 A star connected machine are shown in Figure 4.8 [9]. The machine parameters are $R_1 = 0.10$ P.U., $X_{d1} = 0.112$ P.U., $R_2 = 0.0736$ P.U., $X_{d2} = 0.1$ P.U., and

$$
E_i (P.U.) = 1.345 - 0.203X_m; \quad X_m < 1.728 P.U.
$$

$$
E_i (P.U.) = 1.901 - 0.525X_m; \quad 1.728 \leq X_m \leq 2.259
$$

$$
E_i (P.U.) = 3.156 - 1.08X_m; \quad 2.259 \leq X_m \leq 2.446
$$

$$
E_i (P.U.) = 37.79 - 15.12X_m; \quad 2.446 \leq X_m \leq 2.48.
$$

$$
0; \quad X_m > 2.48
$$

$$
R_m = 18.51 + E_i \times 4.197
$$

The typical collapse of terminal voltage, even at resistive load ($\varphi = 0$), below rated machine current, is evident.

4.4.2 SEIGs with Series Capacitance Compensation

In an attempt to increase the load range (in P.U.), series capacitors are added in short shunt (Figure 4.9a) or long shunt (Figure 4.9b) connections.

FIGURE 4.9 Series compensation by capacitance: (a) short shunt and (b) long shunt.
The short shunt was proven superior in extending the stable operation load range for the same capacitance effort. The investigation of both connections can be done by following the iterative method in the previous paragraph by incorporating \((-jX_s/f)\) in the load (short shunt) or to the stator leakage reactance \(fX_y\) (long shunt). Typical load voltage/current curves with long shunt and short shunt compensation, for the same machine, are shown in Figure 4.10 [9], with \(K = X_s/X_p\) as the ratio between series and parallel capacitor reactances (Figure 4.10a and Figure 4.10b).

Though the voltage collapse was avoided up to rated machine current, the voltage regulation is still noticeable for both connections. Notice that the parallel capacitor \(C_p\) is larger for the long shunt connection \((K = C_s/C_p)\).
**Double Fed Induction Generator-Basic Principles (DFIG)**

Wound rotor induction generators (WRIGs) are provided with three phase windings on the rotor and on the stator. They may be supplied with energy at both rotor and stator terminals. This is why they are called doubly fed induction generators (DFIGs) or double output induction generators (DOIGs). Both motoring and generating operation modes are feasible, provided the power electronics converter that supplies the rotor circuits via slip-rings and brushes is capable of handling power in both directions. As a generator, the WRIG provides constant (or controlled) voltage $V_s$ and frequency $f_1$ power through the stator, while the rotor is supplied through a static power converter at variable voltage $V_r$ and frequency $f_2$. The rotor circuit may absorb or deliver electric power. As the number of poles of both stator and rotor windings is the same, at steady state, according to the frequency theorem, the speed $\omega_m$ is as follows:

$$\omega_m = \omega_1 \pm \omega_2; \quad \omega_m = \Omega_k \cdot p_1$$

(1.1)

where
- $p_1$ is the number of pole pairs
- $\Omega_k$ is the mechanical rotor speed

The sign is positive (+) in Equation 1.1 when the phase sequence in the rotor is the same as in the stator and $\omega_m < \omega_1$, that is, sub synchronous operation. The negative (−) sign in Equation 1.1 corresponds to an inverse phase sequence in the rotor when $\omega_m > \omega_1$, that is, super synchronous operation. For constant frequency output, the rotor frequency $\omega_2$ has to be modified in step with the speed variation. This way, variable speed at constant frequency (and voltage) may be maintained by controlling the voltage, frequency, and phase sequence in the rotor circuit. It may be argued that the WRIG works as a synchronous generator (SG) with three-phase alternating current (AC) excitation at slip (rotor) frequency $\omega_2 = \omega_1 - \omega_m$. However, as $\omega_1 \neq \omega_m$, the stator induces voltages in the rotor circuits even at steady state, which is not the case in conventional SGs. Additional power components thus occur. The main operational modes of WRIG are depicted in Figure 1.1a through Figure 1.1d (basic configuration shown in Figure 1.1a). The first two modes (Figure 1.1b and Figure 1.1c) refer to the already defined sub synchronous and super synchronous generations. For motoring, the reverse is true for the rotor circuit; also, the stator absorbs active power for motoring. The slip $S$ is defined as follows:

www.ravivarmans.com
\[ S = \frac{\omega_2}{\omega_1} \]

> 0; sub synchronous operation

< 0; super synchronous operation

\[ S = \frac{\omega_2}{\omega_1} \]

A WRIG works, in general, for \( \omega_2 \neq 0 \) (\( S \neq 0 \)), the machine retains the characteristics of an induction machine. The main output active power is delivered through the stator, but in super synchronous operation, a good part, about slip stator powers (SPs), is delivered through the rotor circuit. With limited speed variation range, say from \( S_{\text{max}} \) to \(-S_{\text{max}}\), the rotor-side static converter rating — for zero reactive power capability on the rotor side — would be \( P_{\text{CONV}} \approx |S_{\text{max}}| P_s \). With \( S_{\text{max}} \) typically equal to \( \pm 0.2 \) to 0.25, the static power converter ratings and costs would correspond to 20 to 25% of the stator delivered output power. At maximum speed, the WRIG will deliver increased electric power, \( P_{\text{max}} \).
with the WRIG designed at \( P_s \) for \( \omega_m = \omega_1 \) speed. The increased power is delivered at higher than rated speed:

\[
\omega_{\text{max}} = \omega_1 (1 + | S_{\text{max}} |)
\]  \hspace{1cm} (1.4)

Consequently, the WRIG is designed electrically for \( P_s \) at \( \omega_m = \omega_1 \), but mechanically at \( \omega_{\text{max}} \) and \( P_{\text{max}} \). The capability of a WRIG to deliver power at variable speed but at constant voltage and frequency represents an asset in providing more flexibility in power conversion and also better stability in frequency and voltage control in the power systems to which such generators are connected. The reactive power delivery by WRIG depends heavily on the capacity of the rotor-side converter to provide it. When the converter works at unity power delivered on the source side, the reactive power in the machine has to come from the rotor-side converter. However, such a capability is paid for by the increased ratings of the rotor-side converter. As this means increased converter costs, in general, the WRIG is adequate for working at unity power factor at full load on the stator side. Large reactive power releases to the power system are still to be provided by existing SGs or from WRIGs working at synchronism \( (S = 0, \omega_2 = 0) \) with the back-to-back pulse-width modulated (PWM) voltage converters connected to the rotor controlled adequately for the scope. Wind and small hydro energy conversion in units of 1 megawatt (MW) and more per unit require variable speed to tap the maximum of energy reserves and to improve efficiency and stability limits. High-power units in pump-storage hydro-(400 MW) and even thermo power plants with WRIGs provide for extra flexibility for the ever-more stressed distributed power systems of the near future. Even existing (old) SGs may be retrofitted into WRIGs by changing the rotor and its static power converter control. The WRIGs may also be used to generate power solely on the rotor side for rectifier loads (Figure 1.1d). To control the direct voltage (or direct current [DC]) in the load, the stator voltage is controlled, at constant frequency \( \omega_1 \), by a low-cost alternating current (AC) three-phase voltage changer. As the speed increases, the stator voltage has to be reduced to keep constant the current in the DC load connected to the rotor \( (\omega_2 = \omega_1 + \omega_m) \).
If the machine has a large number of poles \((2p_1 = 6, 8, 12)\), the stator AC excitation input power becomes rather low, as most of the output electric power comes from the shaft (through motion). Such a configuration is adequate for brushless exciters needed for synchronous motors (SMs) or for generators, where field current is needed from zero speed, that is, when full-power converters are used in the stator of the respective SMs or SGs. With \(2p_1 = 8\), \(n = 1500\) rpm, and \(f_1 = 50\) Hz, the frequency of the rotor output \(f_2 = f_1 + np_1 = 50 + (1500/60) \times 4 = 150\) Hz. Such a frequency is practical with standard iron core laminations and reduces the contents in harmonics of the output rectified load current.

**Steady-State Equations**

The electromagnetic force (emf) self-induced by the stator winding, with the rotor winding open, \(E_i\) is as follows:

\[
E_i = \pi \sqrt{2} f_1 W_i K_{vi} \phi_{wo}, \quad \text{(RMS)}
\]  \hspace{1cm} (1.20)

\[
K_{vi} = K_{v1} \cdot K_{m1}
\]  \hspace{1cm} (1.21)

The flux per pole \(\phi_{wo}\) is

\[
\phi_{wo} = \frac{2}{\pi} B_{go} r_i
\]  \hspace{1cm} (1.22)

where
- \(l_i\) is the stack length
- \(\tau\) is the pole pitch
- \(D_{so}\) is the stator bore diameter

\(B_{go}\) is the airgap fundamental flux density peak value:

\[
B_{go} = \frac{\mu_0 F_{mo}}{K_{Cq}(1 + K_s)}
\]  \hspace{1cm} (1.23)

\(F_{mo}\) is the amplitude of stator mmf fundamental per pole

From Equation 1.17, with \(n = 1,\)

\[
F_{go} = \frac{3W_1 K_{M1} I_e \sqrt{2}}{\pi p_1}
\]  \hspace{1cm} (1.24)
FIGURE 1.7 Typical airgap flux density ($B_{g(t)}$) and magnetization inductance (in per unit [P.U.]) vs. P.U. stator current.

But the same emf $E_i$ may be expressed as

$$E_i = \omega_i I_{im} - I_{io}$$

(1.25)

So, the main flux, magnetization (cyclic) inductance of the stator — with all three phases active and symmetric — $L_{im}$ is as follows (from Equation 1.20 through Equation 1.25):

$$L_{im} = \frac{6\mu_0}{\pi^2} P^2 K_{g} (1 + K_r)$$

(1.26)

The Carter coefficient $K_c > 1$ accounts for both stator and rotor slot openings ($K_c = K_{c1} K_{c2}$). The saturation factor $K_s$, which accounts for the iron core magnetic reluctance, varies with stator mmf (or current for a given machine), and so does magnetic inductance $L_{im}$ (Figure 1.7).

Besides $L_{im}$, the stator is characterized by the phase resistance $R_s$ and leakage inductance $L_{sl}$ [2]. The same stator current induces an emf $E_{sl}$ in the rotor open-circuit windings. With the rotor at speed $\omega_r$ — slip $S = (\omega_r - \omega_s)/\omega_s$ — $E_{sl}$ has the frequency $f_s = Sf_1$:

$$E_{sl}(t) = E_{sl} \sqrt{2} \cos \omega_2 t$$

$$E_{sl} = \sqrt{2} S f_1 W_1 K_{s1} \omega_s$$

(1.27)

Consequently,

$$\frac{E_{sl}}{E_1} = \frac{S f_1 W \kappa_{s1}}{W f_s} = S K_{s1}$$

(1.28)

This rotor emf at frequency $f_1$ in the rotor circuit is characterized by phase resistance $R_f$ and leakage inductance $L_{r'}$. Also, the rotor is supplied by a system of phase voltages at the same frequency $\omega_o$ and at a prescribed phase.

The stator and rotor equations for steady-state/phase may be written in complex numbers at frequency $\omega_1$ in the stator and $\omega_2$ in the rotor:

$$(R + j\omega L) I_1 - V_1 = E_1 \text{ at } \omega_1$$

(1.29)

$$(R_f + jS \omega L) I_2' - V_2' = E_{sl} \text{ at } \omega_2$$

(1.30)
According to Equation 1.28, we may multiply Equation 1.30 by \(1/K_p\) to reduce the rotor to stator:

\[
(R_s + j\omega L_m)I_s - \frac{V_s}{S} = \frac{E_s}{K_p}; \quad E_s = SE_1K_p
\]

\[
R_s = \frac{R_s}{K_p^2}; \quad L_s = \frac{L_s}{K_p^2}
\]

\[
V_s = \frac{V_m}{K_p}; \quad I_r = \frac{I_r}{K_p}
\]

(1.31)

The division of Equation 1.31 by slip \(S\) yields the following:

\[
\left(\frac{R_s}{S} + j\omega L_m\right)I_s - \frac{V_s}{S} = \frac{SE_1}{S}
\]

(1.32)

But, Equation 1.31 may also be interpreted as being "converted" to frequency \(\omega_1\), as \(E_s\) is at \(\omega_1\) (\(E_{m1}/S = E_1\)):

\[
\left(\frac{R_s}{S} + j\omega L_m\right)I_s - \frac{V_s}{S} = E_1; \quad \text{at } \omega_1
\]

(1.33)

In Equation 1.33, the rotor voltage \(V_s\) and current \(I_r\) vary with the frequency \(\omega_1\) and, thus, are written (in fact) in stator coordinates. A "rotation transformation" has been operated this way. Also, all variables are reduced to the stator. Physically, this would mean that Equation 1.33 refers to a rotor at standstill, which may produce or absorb active power to cover the losses and delivers in motoring the mechanical power of the actual machine it represents.

Finally, the emf \(E_1\) may now be conceived to be produced by both \(I_1\) and \(I_r\) (at the same frequency \(\omega_1\)), both acting upon the magnetization inductance \(L_{m1}\) as the rotor circuit is reduced to the stator:

\[
E_1 = -j\omega_1 L_{m1}(L_s + L_r) = -j\omega_1 L_{m1}L_s
\]

(1.34)

**Equivalent Circuit**

The equivalent circuit corresponding to Equation 1.29, Equation 1.31, and Equation 1.34 is illustrated in Figure 1.8. Two remarks about Figure 1.8 are in order:

- The losses in the machine occur as stator and rotor winding losses \(p_{con} + p_{con}\), core losses \(p_{core}\) and mechanical losses \(p_{mech}\):

\[
\begin{align*}
p_{con} &= 3R_sI_1^2; \\
p_{con} &= 3R_rI_r^2; \\
p_{con} &= 3R_{m1}(S\omega_1)I_1^2
\end{align*}
\]

(1.35)

![Figure 1.8](http://www.ravivarman.com)

FIGURE 1.8 Wound rotor induction generator (WRIG) equivalent circuit for steady state.
The resistance $R_{m}$ that represents the core losses depends slightly on slip frequency $\omega_2 = \omega_2$, as non-negligible core losses also occur in the rotor core for $S_f > 5 \text{ Hz}$.

The active power balance equations are straightforward, from Figure 1.8, as the difference between input electrical powers $P_i$ and $P_o$ and the losses represents the mechanical power $P_m$.

$$P_m = \frac{3 R_{m} I_r^2}{S} - \frac{3 \text{Re}(I_r V_r)}{S} (1 - S) = \frac{T_c}{P_r} (1 - S) = P_{\text{elec}} (1 - S)$$

(1.36)

$$\Sigma P = P_{\text{elec}} + P_{\text{cor}} + P_{\text{mech}} + P_{\text{r}}$$

$P_{\text{elec}}$ is the electromagnetic (through airgap) power.

$$P_i + P_r' = 3 \text{Re}(V_r L_r^+ \omega_2) + 3 \text{Re}(V_r L_r^- \omega_2) = P_m + \Sigma P$$

(1.37)

$T_c$ is the electromagnetic torque. The sign of mechanical power for given motion direction is used to discriminate between motoring and generating. The positive sign (+) of $P_m$ is considered here for motoring (see the association of directions for $V_r, L_r$ in Figure 1.8).

The motor/generator operation mode is determined (Equation 1.36) by two factors: the sign of slip $S$ and the sign and relative value of the active power input (or extracted) electrically from the rotor $P_r'$ (Table 1.1). So, the WRIG may operate as a generator or a motor both sub synchronously ($\omega_0 < \omega_1$) and supersynchronously ($\omega_0 > \omega_1$). The power signs in Table 1.1 may be portrayed as in Figure 1.9.

If all the losses are neglected, from Equation 1.36 and Equation 1.37:

$$P_m = -P_r \frac{(1 - S)}{S} = P_i + P_r$$

(1.38)

Consequently,

$$P_r = -SP_i$$

(1.39)

The higher the slip, the larger the electric power absorption or delivery through the rotor. Also, it should be noted that in supersynchronous operation, both stator and rotor electric powers add up to convert the mechanical power. This way, up to a point, exercising, in terms of torque capability, is not required when operation at $S = -S_{\text{max}}$ occurs with the machine delivering $P_r (1 + |S_{\text{max}}|)$ total electric power.

Reactive power flow is similar. From the equivalent circuit,

$$Q_r + Q_i = 3 \text{Imag}(V_r L_r^+ \omega_2) + 3 \text{Imag}(V_r L_r^- \omega_2) = 3 \omega_2 (L_r I_r^+ + L_r I_r^+ + L_r I_r^+)$$

(1.40)

<table>
<thead>
<tr>
<th>TABLE 1.1 Operation Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
</tr>
<tr>
<td>Subsynchronous ($\omega_0 &lt; \omega_1$)</td>
</tr>
<tr>
<td>Operation Mode</td>
</tr>
<tr>
<td>$P_m$</td>
</tr>
<tr>
<td>$P_r$</td>
</tr>
<tr>
<td>$P_i$</td>
</tr>
</tbody>
</table>
So, the reactive power required to magnetize the machine may be delivered by the rotor or by the stator or by both. The presence of $S$ in Equation 1.40 is justified by the fact that machine magnetization is perceived in the stator at stator frequency $\omega_1$. As the static power converter rating depends on its rated apparent power rather than active power, it seems to be practical to magnetize the machine from the stator. In this case, however, the WRIG absorbs reactive power through the stator from the power grids or from a capacitive-resistive load. In stand-alone operation mode, however, the WRIG has to provide for the reactive power required by the load up to the rated lagging power factor conditions. If the stator operates at unity power factor, the rotor-side static power converter has to deliver reactive power extracted either from inside itself (from the capacitor in the DC link) or from the power grid that supplies it. As magnetization is achieved with lowest kVAR in DC, when active power is not needed, the machine may be operated at synchronism ($\omega_r = \omega_1$) to fully contribute to the voltage stability and control in the power system. To further understand the active and reactive power flows in the WRIG, phasor diagrams are used.
DYNAMIC MODEL OF PM SYNCHRONOUS MOTORS

ABSTRACT: In a permanent magnet synchronous motor where inductances vary as a function of rotor angle, the 2 phase (d-q) equivalent circuit model is commonly used for simplicity and intuition. In this article, a two phase model for a PM synchronous motor is derived and the properties of the circuits and variables are discussed in relation to the physical 3 phase entities. Moreover, the paper suggests methods of obtaining complete model parameters from simple laboratory tests. Due to the lack of developed procedures in the past, obtaining model parameters were very difficult and uncertain, because some model parameters are not directly measurable and vary depending on the operating conditions. Formulation is mainly for interior permanent magnet synchronous motors but can also be applied to surface permanent magnet motors.

I. INTRODUCTION

PM synchronous motors are now very popular in a wide variety of industrial applications. A large majority of them are constructed with the permanent magnets mounted on the periphery of the rotor core. Following [1], we will call them as the Surface Permanent Magnet (SPM) synchronous motors. When permanent magnets are buried inside the rotor core rather than bonded on the rotor surface, the motor not only provides mechanical ruggedness but also opens a possibility of increasing its torque capability. By designing a rotor magnetic circuit such that the inductance varies as a function of rotor angle, the reluctance torque can be produced in addition to the mutual reaction torque of synchronous motors. This class of Interior PM (IPM) synchronous motors can be considered as the reluctance synchronous motor and the PM synchronous motor combined in one unit. It is now very popular in industrial and military applications by providing high power density and high efficiency compared to other types of motors.

Conventionally, a 2-phase equivalent circuit model (d-q model) [2] has been used to analyze reluctance synchronous machines. The theory is now applied in analysis of other types of motors [3-7] including PM synchronous motors, induction motors etc. In Section II, an equivalent 2-phase circuit model of a 3-phase IPM machines is derived in order to clarify the concept of the transformation and the relation between 3-phase quantities and their equivalent 2-phase quantities. Although the above equivalent circuit is very popular, discussions on obtaining parameters of the equivalent circuit for a given motor are rarely found. The main objective of the article is to establish a method to obtain 2-phase circuit parameters from physically measured data. Throughout the article, the following assumptions are made:

1. Stator windings produce sinusoidal mmf distribution. Space harmonics in the air-gaps are neglected.
2. Air-gap reluctance has a constant component as well as a sinusoidally varying component.
3. Balanced 3 phase supply voltage is considered.
4. Although magnetic saturation is considered, eddy current and hysteresis effects are neglected.

In addition, presence of damper windings are not considered here because they are not used in PM synchronous machines in general. A model with short-circuited damper windings may be used to analyze eddy current effects. Nomenclature on this article is listed in the following. More definitions may appear as appropriate during discussion. For notational convenience, units on all angles are in degrees.

P: number of poles of the motor.
Ia, Ib, Ic: phase a, b, c instantaneous stator current.
Va, Vb, Vc: phase a, b, c instantaneous stator voltage.
Id, Iq: d- and q-axis components of stator current.

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Vd, Vq: d and q-axis components of stator phase voltage
Rs: stator resistance
p: d/dt
Ld, Lq: d- and q-axis stator self inductance
Ls: Average inductance. \( Ls = \frac{1}{2} (Lq + Ld) \)
Lx: Inductance fluctuation. \( Lx = \frac{1}{2} (Lq - Ld) \)
\( \lambda m \): peak flux linkage due to permanent magnet.
\( \theta \): electrical angle between a-axis and q-axis in degrees. See Fig. 2.1.
\( \psi \): \( \theta = p \psi \), angular velocity of rotation (in electrical rad/Sec.)

II. DERIVATION OF A PM SYNCHRONOUS MOTOR MODEL

Fig. 2.1 illustrates a conceptual cross-sectional view of a 3-phase, 2-pole IPM synchronous motor along with two reference frames. To illustrate the inductance difference (Lq > Ld), rotor is drawn with saliency although actual rotor structure is more likely a cylinder. The stator reference axis for the a-phase is chosen to the direction of maximum mmf when a positive a-phase current is supplied at its maximum level. Reference axis for b- and c- stator frame are chosen 120° and 240° (electrical angle) ahead of the a-axis, respectively. Following the convention of choosing the rotor reference frame, the direction of permanent magnet flux is chosen as the d-axis, while the q-axis is 90 degrees ahead of the d-axis. The angle of the rotor q-axis with respect to the stator a-axis is defined as \( \theta \). Note that as the machine turns, the d-q reference frame is rotating at a speed of \( \theta = \frac{d\theta}{dt} \), while the stator a-, b-, c- axes are fixed in space. We will find out later that the choice of this rotating frame greatly simplifies the dynamic equations of the model.

The electrical dynamic equation in terms of phase variables can be written as

\[
V_a = R_s I_a + p \lambda_a
\]
\[
V_b = R_s I_b + p \lambda_b
\]
\[
V_c = R_s I_c + p \lambda_c
\]

while the flux linkage equations are

\[
\lambda_a = L_{aa} I_a + L_{ab} I_b + L_{ac} I_c + \lambda_{ma}
\]
\[
\lambda_b = L_{ba} I_a + L_{bb} I_b + L_{bc} I_c + \lambda_{mb}
\]
\[
\lambda_c = L_{ca} I_a + L_{cb} I_b + L_{cc} I_c + \lambda_{mc}
\]

considering symmetry of mutual inductances such as \( L_{ab} = L_{ba} \). Note that in the above equations, inductances are functions of the angle \( \theta \). Since stator self inductances are maximum when the rotor q-axis is aligned with the phase, while mutual inductances are maximum when the rotor q-axis is in the midway between two phases. Also, note that the effects of saliency appeared in stator self and mutual inductances are indicated by the term 2\( \theta \).

\[
L_{aa} = L_{so} + L_{sl} + L_x \cos (2\theta)
\]
\[
L_{bb} = L_{so} + L_{sl} + L_x \cos (2\theta + 120)
\]
\[
L_{cc} = L_{so} + L_{sl} + L_x \cos (2\theta - 120)
\]
\[
L_{ab} = -(1/2) L_{so} + L_x \cos (2\theta - 120)
\]
\[
L_{bc} = -(1/2) L_{so} + L_x \cos (2\theta)
\]
\[
L_{ac} = -(1/2) L_{so} + L_x \cos (2\theta + 120)
\]

For mutual inductances in the above equations, the coefficient -(1/2) comes due to the fact that stator phases are displaced by 120°, and \( \cos(120) = -(1/2) \). Meanwhile, flux-linkages at the stator windings due to the permanent magnet are
\[ \lambda_{ma} = \lambda_m \cos(\theta) \quad (2.13) \]
\[ \lambda_{mb} = \lambda_m \cos(\theta - 120) \quad (2.14) \]
\[ \lambda_{mc} = \lambda_m \cos(\theta + 120) \quad (2.15) \]

For this model, input power \( P_i \) can be represented as
\[ P_i = V_a I_a + V_b I_b + V_c I_c \quad (2.16) \]

Unfortunately, the output power \( P_o \) and the output torque \( T = \frac{P}{2} \frac{P_o}{\lambda} \) cannot simply be derived in this 3-phase model. The torque can be expressed as
\[ T = \left( \frac{P}{6} \right) (L_q - L_d) \left[ \{ (I_a^2 - 0.5 I_b^2 - 0.5 I_c^2 - I_a I_b - I_a I_c + 2 I_b I_c) \sin 2\theta \\
+ (\frac{3}{2}) (I_b^2 + I_c^2 - 2 I_a I_b + 2 I_a I_c) \cos 2\theta + \lambda_m \{ (I_a - 0.5 I_b - 0.5 I_c) \cos \theta \\
+ (\frac{3}{2}) (I_b - I_c) \sin \theta \} \right]. \quad (2.17) \]

Refer to [6] for detailed derivation by using energy method.

![Fig. 2.1 PM Synchronous Motor](www.ravivarman.com)

Now, let \( S \) represent any of the variables (current, voltage, and flux linkage) to be transformed from the \( a-b-c \) frame to \( d-q \) frame. The transformation in matrix form is given by
\[ \begin{bmatrix} S_q \\ S_d \\ S_o \\ S_c \end{bmatrix} = \frac{2}{3} \begin{bmatrix} \cos \theta & \cos(\theta - 120) & \cos(\theta + 120) \\ \sin \theta & \sin(\theta - 120) & \sin(\theta + 120) \\ 0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} S_a \\ S_b \\ S_c \end{bmatrix} \quad (2.18) \]

Here \( S_o \) component is called the zero sequence component, and under balanced 3-phase system this component is always zero. Since it is a linear transformation, its inverse transformation exists and is
\[ \begin{bmatrix} S_a \\ S_b \\ S_c \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 1 \\ \cos(\theta - 120) & \sin(\theta - 120) & 1 \\ \cos(\theta + 120) & \sin(\theta + 120) & 1 \end{bmatrix} \begin{bmatrix} S_q \\ S_d \\ S_o \end{bmatrix} \quad (2.19) \]

Now, by applying the transform of Eq. 2.18 to voltages, flux-linkages and currents of Eqs.2.1-2.6, we get a set of simple equations as
\[ V_q = R_s I_q + \lambda \dot{q} + \omega \lambda_d \quad (2.20) \]
\[ V_d = R_s I_d + p \lambda_d - (\omega) \lambda_q \] (2.21)

where

\[ \lambda_q = L_q I_q \] (2.22)
\[ \lambda_d = L_d I_d + \lambda_m \] (2.23)

Here, \( L_q \) and \( L_d \) are called d- and q-axis synchronous inductances, respectively, and are defined as

\[ L_q = (3/2) (L_{so} + L_x) + L_{sl} \] (2.24)
\[ L_d = (3/2) (L_{so} - L_x) + L_{sl} \] (2.25)

As noticed in the above equation, synchronous inductances are effective inductances under balanced 3 phase conditions. Each synchronous inductance is made up of self inductance (which includes leakage inductance) and contributions from other 2 phase currents. Now, a more convenient equation may result by eliminating the flux-linkage terms from Eqs.2.20-2.21 as

\[ V_q = (R_s + L_q p) I_q + L_d I_d + \lambda_m \] (2.26)
\[ V_d = (R_s + L_d p) I_d - L_q I_q \] (2.27)

Fig. 2.1 shows a dynamic equivalent circuit of an IPM synchronous machine based on Eqs.2.26-2.27. Note that in practice, magnetic circuits are subject to saturation as current increases. Especially, when \( I_q \) is increased, the value of \( L_q \) is decreased and \( \lambda_m \) and \( L_d \) is subject to armature reaction. Since \( I_d \) is maintained to zero or negative value (demagnetizing) in most operating conditions, saturation of \( L_d \) rarely occurs.

Fig. 2.1 Equivalent Circuit of a PM Synchronous Motor

For this model, instantaneous power can be derived from Eq. 2.16 via transformation as

\[ P_i = (3/2) \{ V_q I_q + V_d I_d \} \] (2.28)

neglecting the zero sequence quantities. The output power can be obtained by replacing \( V_q \) and \( V_d \) by the associated speed voltages as

\[ P_o = (3/2) \{ - (\omega) \lambda_q I_d + (\omega) \lambda_d I_q \} \] (2.29)

The produced torque \( T \), which is power divided by mechanical speed can be represented as

\[ T = (3/2) (P/2) (\lambda_m I_q + (L_d - L_q) I_q I_d) \]. (2.30)

It is apparent from the above equation that the produced torque is composed of two distinct mechanisms. The first term corresponds to "the mutual reaction torque" occurring between \( I_q \) and the permanent magnet, while the second term corresponds to "the reluctance torque" due to the differences in d-axis and q-axis reluctance (or inductance). Note that in order to produce additive reluctance torque, \( I_d \) must be negative since \( L_q > L_d \).
In order to discuss about the relation between the original 3-phase system and the 2-phase equivalent system, consider the transformation of Eq. 2.18-2.19. In fact, this reversible linear transformation can be interpreted as a combination of two transformations. First, let $(\alpha|\beta)$ axis be a stationary frame so that $\alpha$-axis and $\beta$-axis coincide with q- and d-axis, respectively when the angle $0$ is zero. The transformation from 3-axis variables to 2-axis variables is

$$
\begin{bmatrix}
S\alpha \\
S\beta \\
So
\end{bmatrix} = (2/3)
\begin{bmatrix}
1 & \cos (120^\circ) & \cos (120^\circ) \\
0 & -\sin (120^\circ) & \sin (120^\circ) \\
0.5 & 0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
Sa \\
Sb \\
Sc
\end{bmatrix} \tag{2.31}
$$

As before, the variable $S$ represents voltage, current or flux linkage and the zero sequence component $So$ is always zero for a balanced 3 phase system. The second transformation simply converts from stationary $\alpha$-$\beta$ frame to the rotating d-q frame as

$$
\begin{bmatrix}
Sd \\
Sq \\
So
\end{bmatrix} =
\begin{bmatrix}
\cos 0 & -\sin 0 & 0 \\
\sin 0 & \cos 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
S\alpha \\
S\beta \\
So
\end{bmatrix} \tag{2.32}
$$

It can be easily verified that multiplication of the above two transformations lead to Eq. 2.18. Since the d-q frame itself is rotating at a synchronous frequency, all sinusoidally modulated terms vanish after the transformation into this d-q frame. Notice that since the transformation of Eq. 2.18 (and Eq. 2.31) is not unitary (A square matrix is unitary if its inverse is the same as its transpose), the power and torque of the 2-phase equivalent system is different from those of the original 3-phase system. To calculate power and torque from the 2-phase equivalent circuit, (3/2) factor has to be included as shown in Eq. 2.28-2.30. The reason that this non-unitary transformation of Eq. 2.18 is popular is because the magnitude of voltages, currents and flux linkages are the same in both frames. See Appendix for further information regarding a unitary transformation. In this 2-phase equivalent circuit, inductances are roughly (3/2) times those of the actual 3 phase value (refer to Eq. 2.24-25). These "synchronous inductances" $Ld$ and $Lq$ are the effective inductances seen by a phase winding during balanced operation. In addition to the self-flux linkage of one phase, additional flux linkages are produced by appropriately adjusting the angle $\phi$m.

For some applications, it is useful to define voltage vector $Vs$ and current vector $Is$ whose magnitudes are

$$
|Vs| = Vs = \sqrt{Vq^2 + Vd^2}, \tag{2.33}
$$

$$
|Is| = Is = \sqrt{Iq^2 + Id^2}. \tag{2.34}
$$

Assume that current vector $Is$ is $\phi$m degrees ahead of the q-axis. Then, the relation between the stator current magnitude $Is$, and $Id$ and $Iq$ are

$$
Iq = Is \cos \phi m, \tag{2.35}
$$

$$
Id = -Is \sin \phi m, \tag{2.36}
$$

and Eq. 2.30 can be expressed in terms of $\phi m$ as

$$
T = (3/2)(P/2)\phi m Is \cos \phi m + 0.5 (Lq-Ld) Is^2 \sin 2\phi m. \tag{2.37}
$$

For surface PM motors whose $Lq = Ld$, the reluctance torque term of the above equation vanishes and the above equation is reduced to

$$
T = (3/2) (P/2) \phi m Is \cos \phi m. \tag{2.38}
$$

Here, the maximum torque is produced when $\phi m = 0^\circ$, or the angle between the stator flux linkage vector and the PM flux linkage vector on the rotor is 90 degrees which is analogous to the characteristics of a separately excited DC motor. For interior PM synchronous motors, the reluctance torque is not negligible and higher torque can be produced by appropriately adjusting the angle $\phi m$. 

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